1 Nuts and bolts

1. Visit my office by the end of the week. Office hours this week: MW 11-12, Th 12-1, or by appointment.

2. Before workshop tomorrow, read the following sections of the text: 6315 Definition of derivative, 6316 The power formula.

3. Added note:
   (a) Work example 3 for Monday.
   (b) Experiment with example 4 by thinking of $x^6$ as $x^3 \cdot x$, $x^3 \cdot x^3$, etc.
   (c) We did not introduce the exponential function and its derivative, but I have left those slides in. Ask your workshop leader to discuss this before you work on the related problem in workshop Thursday.

2 The main point from Lecture 2 on Monday

The concept of the derivative arises naturally in (at least) two ways: in finding the slope of a tangent line to a curve, and in finding the instantaneous velocity of a moving object, given a function describing its position.

3 What’s happening today

1. Finish velocity problem from Monday

2. Summarize by writing the definition of derivative

3. Finding derivatives using the definition

4. Now for some shortcuts: power rule, polynomials, product rule, exponential function
4 The velocity problem

Example 1. Suppose that you are standing on a 24 foot platform. You throw a ball into the air with an initial vertical velocity of 16 feet per second. The ball rises into the air and then falls and hits the ground. Suppose the height of the ball at time $t$ seconds after you throw it is given by

$$h(t) = -16t^2 + 16t + 24.$$

a. How long does it take for the ball to hit the ground?

b. When is the ball again even with the platform?

c. What is the average velocity of the ball between the (second) time it is even with the platform and the time it hits the ground?

d. What is the velocity of the ball at the instant it is even with the platform?

5 Definition of the derivative

Given a function $f(x)$, the derivative of $f$ at $x = b$ is defined to be

$$f'(b) = \lim_{x \to b} \frac{f(x) - f(b)}{x - b},$$
if the limit exists.

Equivalently, it is defined to be

\[ f'(b) = \lim_{{h \to 0}} \frac{f(b + h) - f(b)}{h}, \]

if the limit exists.

**Example 2.** Find \( f'(b) \) for \( b \neq 0 \) when \( f(x) = \frac{1}{x} \).

Use this definition:

\[ f'(b) = \lim_{{x \to b}} \frac{f(x) - f(b)}{x - b}. \]

Thus, \( f'(b) = -\frac{1}{b^2} \).

We can also write \( f'(x) = -\frac{1}{x^2} \) if we want to emphasize that the derivative is also a function of \( x \).

Note that \( f(x) = x^{-1} \), and \( f'(x) = -\frac{1}{x^2} = (-1)x^{-2} \). This is an example of the power rule, our first “shortcut”.

3
6 The power rule

Let \( f(x) = x^n \), where \( n \) is a positive integer \( (n = 1, 2, 3, \ldots) \). Then \( f'(x) = nx^{n-1} \).

Think “bring the power down in front and raise \( x \) to the power that is one smaller”.

Note: The power rule works also for any real number \( n \neq 0 \).

But why is this true? The reason requires the binomial theorem that we mentioned Monday.

7 The nitty-gritty

Let’s use the second definition of derivative. Let \( f(x) = x^n \), where \( n \) is a positive integer.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}.
\]

Now, how do we deal with \((x + h)^n\)?

The \( n \)th row of Pascal’s triangle gives the coefficients of the expansion of \((x + h)^n\).

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
& & & & & & & \\
1 & n & & & & \cdots & n & 1 \\
& & & & & & & \\
\end{array}
\]

So \((x + h)^n = x^n + nx^{n-1}h + \text{a bunch of other terms}\) (that involve higher powers of \( h \)).

Then

\[
\begin{align*}
\frac{f(x + h) - f(x)}{h} &= \frac{(x + h)^n - x^n}{h} \\
&= \frac{x^n + nx^{n-1}h + \text{a bunch of other terms} - x^n}{h} \\
&= nx^{n-1} + \text{a different bunch of other terms}.
\end{align*}
\]

As \( h \) tends to 0, this quantity tends to \( nx^{n-1} \).
8 Differentiating polynomials

We can differentiate polynomials easily with the power rule along with the following three rules:

- **Note:** the derivative $f'(x)$ of a function $f(x)$ is sometimes written $\frac{df}{dx} f(x)$.

- **Constants** If $f(x) = c$, where $c$ is a constant, then $f'(x) = 0$.

- **Constant times function** Given a function $f(x)$ and a constant $c$, we have

  \[ \frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x). \]

  "You can pull the constant out in front".

- **Sums of functions** Given functions $f(x)$ and $g(x)$, we have

  \[ \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x). \]

  \[ \text{Example 3. Find } \frac{d}{dx}(\pi x^3 + 3x^\pi - \frac{4}{3}x^\frac{1}{4}). \]

  We saw on the last slide that the derivative of a sum of functions is the sum of the derivatives. We really wish it were true that the derivative of a product of functions is the product of the derivatives.

  **THIS IS NOT THE PRODUCT RULE. THIS IS A FALSE STATEMENT.** If $f(x)$ and $g(x)$ are functions, then

  \[ \frac{d}{dx}(f(x) \cdot g(x)) = (\frac{d}{dx} f(x)) \cdot (\frac{d}{dx} g(x)). \]

  Example to show this is false?

9 The product rule

If $f(x)$ and $g(x)$ are functions, then

\[ \frac{d}{dx}(f(x) \cdot g(x)) = (\frac{d}{dx} f(x)) \cdot g(x) + f(x) \cdot (\frac{d}{dx} g(x)). \]

**Example 4.** Find the derivative of $x^6$ using the power rule and then again by thinking of $x^6$ as $x^4 \cdot x^2$. 

10 Exponential functions

Investigate the following limit numerically.

\[
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n
\]

\[\begin{array}{|c|c|} 
\hline
n & \left(1 + \frac{1}{n}\right)^n \\
\hline
1 & 2 \\
2 & 2.25 \\
4 & 2.4414\ldots \\
10 & 2.5937\ldots \\
100 & 2.7048\ldots \\
1000 & 2.7169\ldots \\
10000 & 2.7181\ldots \\
100000 & 2.7182\ldots \\
\hline
\end{array}
\]

It turns out that \( (1 + \frac{1}{n})^n \) tends to the number \( e \approx 2.71828182846\ldots \). The function \( f(x) = e^x \) is very important in applications, and it is the only function whose derivative is equal to itself:

If \( f(x) = e^x \), then \( f'(x) = e^x \).

**Example 5.** Suppose that \( g(x) \) is a function with \( g(1) = 4 \) and \( g'(1) = -2 \). Let \( f(x) = e^x g(x) \). Find \( f'(1) \).