MATH 2373.020 – Lecture 11 – 2/25/09

HW6, due Tuesday, March 3:
7616 #1-5
7626 #1,2
7632 #1
7634 #1
7638 #1,2

revised Friday office hour: 2:00 - 3:15 instead of 10:30 - 11:45

Today:

- Second-order DE with constant coefficients
- mass spring problems
- LRC circuit problems
- phase angle
- autonomous DE

Summary: Associated to each second-order linear homogeneous DE with constant coefficients is an *auxiliary equation*.

- If the auxiliary equation has distinct real roots m_1 and m_2, then the general solution is $C_1e^{m_1t} + C_2e^{m_2t}$.

- If the auxiliary equation has a repeated real root m, then the general solution is $C_1e^{mt} + C_2te^{mt}$.

- If the auxiliary equation has complex roots $a \pm bi$, then the general solution is $C_1e^{at}\cos(bt) + C_2e^{at}\sin(bt)$.

Spring-mass problems

Example 1. (Example 4 Monday) Suppose that a mass of 1 kg is hanging from a spring with spring constant 4 kg/sec^2. Suppose that there is no external force acting on the spring. Suppose that the spring is pulled to a point 0.1 meters below equilibrium and at $t = 0$ is let go. Find an expression for the movement of the spring under the following assumptions:

a. there is no damping force (vacuum)

b. the damping constant is 2 kg/sec

c. the damping constant is 4 kg/sec

d. the damping constant is 8 kg/sec

Summary: If there is no external force, then a spring-mass system leads to a homogeneous second-order DE with constant coefficients.

- If the auxiliary equation has distinct real roots m_1 and m_2, then we say the system is *overdamped*.

- If the auxiliary equation has a repeated real root m, then we say the system is *critically damped*.

- If the auxiliary equation has complex roots $a \pm bi$, then we say the system is *underdamped*.

LRC series circuit

The charge $Q(t)$ in coulombs on the capacitor is described by this (generally *nonhomogeneous*) second-order linear DE with constant coefficients:

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

where L is the inductance in henries, R is the resistance in ohms, C is the capacitance in farads, and $E(t)$ is the applied voltage in volts. (Also, $Q'(t) = I(t)$ is the current in amperes.)

Example 2. (p.161 #1) $R = .6$ ohms, $C = 1/3$ farad, $L = .3$ henries, $E(t) = 6\cos(25t)$, $Q(0) = 20$ coulombs, $I(0) = Q'(0) = 4$ amperes.

Phase angle

Solutions of second-order linear DE often yield expressions of the form

$$C_1 \cos bt + C_2 \sin bt,$$

and these are easier to analyze if we write it in the form

$$A \sin(wt + d)$$

or

$$\cos(wt - d).$$

Example 3. Rewrite the function

$$5 \cos 2t + 12 \sin 2t$$

as a single sine function and then as a single cosine function.

We use the trigonometric identities

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

and

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

Autonomous DE

A first-order DE is *autonomous* if it has the form

$$\frac{dy}{dt} = f(y).$$

That is, the right-hand-side is independent of t. These are all separable.

Logistic equations are of this form.

Suppose that $y(t) = C$ is an equilibrium solution. If

$$\lim_{t \to +\infty} y(t) = C$$

for all solutions $y(t)$ starting from an initial point sufficiently near C, then we say that the equilibrium is *stable*.