Summary: Associated to each second-order linear homogeneous DE with constant coefficients is an auxiliary equation.

- If the auxiliary equation has distinct real roots $m_1$ and $m_2$, then the general solution is $C_1 e^{m_1 t} + C_2 e^{m_2 t}$.

- If the auxiliary equation has a repeated real root $m$, then the general solution is $C_1 e^{mt} + C_2 te^{mt}$.

- If the auxiliary equation has complex roots $a \pm bi$, then the general solution is $C_1 e^{at} \cos bt + C_2 e^{at} \sin bt$.

Summary: If there is no external force, then a spring-mass system leads to a homogeneous second-order DE with constant coefficients.

- If the auxiliary equation has distinct real roots $m_1$ and $m_2$, then we say the system is overdamped.

- If the auxiliary equation has a repeated real root $m$, then we say the system is critically damped.

- If the auxiliary equation has complex roots $a \pm bi$, then we say the system is underdamped.

Spring-mass problems

Example 1. (Example 4 Monday) Suppose that a mass of 1 kg is hanging from a spring with spring constant 4 kg/sec$^2$. Suppose that there is no external force acting on the spring. Suppose that the spring is pulled to a point 0.1 meters below equilibrium and at $t = 0$ is let go. Find an expression for the movement of the spring under the following assumptions:

a. there is no damping force (vacuum)

b. the damping constant is 2 kg/sec

c. the damping constant is 4 kg/sec

d. the damping constant is 8 kg/sec

LRC series circuit

The charge $Q(t)$ in coulombs on the capacitor is described by this (generally nonhomogeneous) second-order linear DE with constant coefficients:

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

where $L$ is the inductance in henries, $R$ is the resistance in ohms, $C$ is the capacitance in farads, and $E(t)$ is the applied voltage in volts. (Also, $Q'(t) = I(t)$ is the current in amperes.)

Example 2. (p.161 #1) $R = .6$ ohms, $C = 1/3$ farad, $L = .3$ henries, $E(t) = 6 \cos(25t)$, $Q(0) = 20$ coulombs, $I(0) = Q'(0) = 4$ amperes.
**Phase angle**

Solutions of second-order linear DE often yield expressions of the form

\[ C_1 \cos \beta t + C_2 \sin \beta t, \]

and these are easier to analyze if we write it in the form

\[ A \sin(\omega t + d) \]

or

\[ \cos(\omega t - d). \]

**Example 3.** Rewrite the function

\[ 5 \cos 2t + 12 \sin 2t \]

as a single sine function and then as a single cosine function.

We use the trigonometric identities

\[ \sin(x + y) = \sin x \cos y + \cos x \sin y \]

and

\[ \cos(x + y) = \cos x \cos y - \sin x \sin y. \]

**Autonomous DE**

A first-order DE is *autonomous* if it has the form

\[ \frac{dy}{dt} = f(y). \]

That is, the right-hand-side is independent of \( t \). These are all separable.

Logistic equations are of this form.

Suppose that \( y(t) = C \) is an equilibrium solution. If

\[ \lim_{t \to +\infty} y(t) = C \]

for all solutions \( y(t) \) starting from an initial point sufficiently near \( C \), then we say that the equilibrium is *stable*. 