Today:

- eigenvalues and systems of first-order differential equations
- double eigenvalues and their eigenvectors
- “reducing order” using systems of first-order DE
- Introduction to the Laplace transform

Example 1. Find the general solution of the system of differential equations
\[
\begin{align*}
\frac{dx}{dt} &= 2x + 3y \\
\frac{dy}{dt} &= 2x + y.
\end{align*}
\]

Summary: when coefficient matrix has two different real eigenvalues

The general solution of the system \( X' = AX \) where \( X = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \) is

\[
X = c_1 V e^{\lambda_1 t} + c_2 W e^{\lambda_2 t},
\]

where \( V \) and \( W \) are eigenvectors corresponding, respectively, to the different real eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of the matrix \( A \).

Example 2. Solve the initial value problem
\[
\begin{align*}
x' &= x + 2y \\
y' &= 4x + 3y,
\end{align*}
\]
with initial conditions \( x(0) = 1 \) and \( y(0) = 8 \).

We will consider later the cases where the coefficient matrix of the system has

- complex eigenvalues
- a double eigenvalue

For now, let’s work (for the first time) with a matrix that has a double eigenvalue:

Example 3. Find eigenvalues and eigenvectors for the matrix
\[
\begin{pmatrix}
1 & -2 & 2 \\
-2 & 1 & -2 \\
2 & -2 & 1
\end{pmatrix}
\]
Example 3. Note: a double eigenvalue is not guaranteed to have two independent eigenvectors. Consider
\[
\begin{pmatrix}
2 & 1 \\
0 & 2
\end{pmatrix}
\]

Example 4. How did we solve this second-order differential equation?
\[
y'' + 7y' + 12y = 0
\]
Rewrite as a first-order system to produce another approach to solving.

The Laplace transform

1. What is a transform?
An operation that changes one function into another.

2. Why do we need it?
We’ve worked with second-order non-homogeneous DE like this:
\[ay'' + by' + cy = f(t).\]
The function \(f(t)\) may not even be continuous. Laplace transforms help us solve problems of that type.

The Laplace transform

Given a function \(f(t)\) defined for \(t \geq 0\), the Laplace transform of \(f(t)\) is defined to be
\[
\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)\,dt,
\]
when the integral exists.

Example 5. Let
\[
f(t) = \begin{cases} 
2 & \text{for } 3 \leq t \leq 6; \\
0 & \text{for } 0 \leq t < 3 \text{ and for } t > 6.
\end{cases}
\]
Find \(\mathcal{L}\{f(t)\}\).

Try:
\[
\mathcal{L}\{C\}, \text{ that is, } f(t) = C, \text{ a constant.}
\]
\[
\mathcal{L}\{t\}
\]
\[
\mathcal{L}\{\sin t\}\]