1. (20 points) Find an equation for the plane containing the line parametrized by \((x, y, z) = c_1(t) = (2 + t, 2t, 1 - t)\) and the line parametrized by \((x, y, z) = c_2(t) = (3, t - 1, 2t - 6)\). Write your answer in the form \(z = Ax + By + C\).

Solution: The answer is \(z = -5x + 2y + 11\).

We get this using the equation for the plane

\[0 = \langle A, B, C \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle,\]

where \(\langle A, B, C \rangle\) is the normal vector, and \((x_0, y_0, z_0)\) is a point on the plane.

Any point on one of the lines is also a point on the plane, so we can use \(c_1(0) = (2, 0, 1)\) as \((x_0, y_0, z_0)\). The simplest way to get the normal vector is to realize that the coefficients of the \(t\)'s in the equations for \(c_1\) and \(c_2\) give vectors that are parallel to the plane, so there cross product will be the normal vector to the plane. That is, we have two vectors

\[v_1 = \langle 1, 2, -1 \rangle\]
\[v_2 = \langle 0, 1, 2 \rangle.\]

The normal vector is then their cross product,

\[
\begin{vmatrix}
i & j & k \\
1 & 2 & -1 \\
0 & 1 & 2
\end{vmatrix},
\]

which is

\[= \langle 2 \cdot 2 - (-1) \cdot 1, -(1 \cdot 2), 1 \cdot 1 - 2 \cdot 0 \rangle = \langle 5, -2, 1 \rangle.\]

So the equation of the plane is

\[\langle 5, -2, 1 \rangle \cdot \langle x - 2, y, z - 1 \rangle = 0\]

which becomes

\[5x - 10 - 2y + z - 1 = 0\]

which is

\[z = -5x + 2y + 11.\]

Grading Scheme:

Basically, 6 points for valid vectors, 2 points for a valid point on the plane, 4 points for the cross product (both knowing to use it and using it correctly), 4 points for using a valid equation for the plane, 4 points for the correct answer.
2. (30 points) Let \( g(p, s) \) be a function that gives your grade on exams for a course as a function \( p \), the number of hours partying, and \( s \), the number of hours studying each week. For your first exam, you partied 20 hours and studied 15 hours each week, and you received a 2.0, which is a C. Hence, \( g(20, 15) = 2 \). For your second exam, you slightly increased your studying and discovered that \( \frac{\partial g}{\partial s}(20, 15) = 0.3 \). For your third exam, you slightly increased your partying and discovered that \( \frac{\partial g}{\partial p}(20, 15) = -0.5 \). We assume that \( g(p, s) \) is a differentiable function.

(i) (12 points) What is the directional derivative \( D_u g(20, 15) \) where \( u \) is the unit vector pointing in the direction of \((-1, -1)\)? Interpret your answer in terms of the effect of partying and studying on your grade.

**Solution:** Since \( u \) is a unit vector, we need to normalize \((-1, -1)\), i.e.,

\[
 u = \frac{(-1, -1)}{\sqrt{(-1)^2 + (-1)^2}} = \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).
\]

Also,

\[
 \nabla g(20, 15) = \left( \frac{\partial g}{\partial p}(20, 15), \frac{\partial g}{\partial s}(20, 15) \right) = (-0.5, 0.3).
\]

Hence,

\[
 D_u g(20, 15) = \nabla g(20, 15) \cdot u = \langle -0.5, 0.3 \rangle \cdot \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = \frac{\sqrt{2}}{10}.
\]

This means that if you slightly decrease both partying and studying by the same amount, your grade increases at rate \( \frac{\sqrt{2}}{10} \).

I assigned 3 points for normalizing \( u \), 3 points for "correct use" of partial derivatives, 2 points for knowing the definition of the directional derivative, 2 points for calculation, and 2 points for interpretation of the result.

(ii) (13 points) For your final exam, you want to slightly increase your partying and adjust your studying to maintain a grade of 2.0. For every hour of extra partying how much should you increase or decrease your studying so that your grade stays constant at 2.0?

**Solution:** Since the gradient is perpendicular to the level curve at \((20, 15)\), the direction must be perpendicular to \((-0.5, 0.3)\). Let \( a \) denote number of hours of studying you need to increase or decrease for every hour of extra partying to maintain your current grade. Then

\[
 (-0.5, 0.3) \cdot (1, a) = 0.
\]

Solving for \( a \), we get \( a = \frac{2}{3} \) hours or 100 minutes.

I gave 10 points for knowing that you need to move orthogonal to the gradient and setting up the equation correctly, and 3 points for calculation.
(Problem 2 continued)

(iii) (5 points) Below are four possible level curve plots of \( g(p, s) \) (with \( p \) on the \( x \)-axis and \( s \) on the \( y \)-axis). Which of the following could be the level curve plot of \( g(p, s) \)? (Be sure to explain your answer.)

![Graph B](image)

**Solution:** Graph B could be the level curve plot of \( g(p, s) \) because only in B, \((-0.5, 0.3)\) seems perpendicular to the level curve at \((20,15)\).

I gave 2 points for choosing B, and 3 points for explanation.

3. (20 points) Let \( g(p, s) \) be the function from problem 2 that gives your grade on exams as a function \( p \), the number of hours partying, and \( s \), the number of hours studying each week. Recall that \( g(20, 15) = 2 \), \( \frac{\partial g}{\partial p}(20, 15) = 0.3 \), and \( \frac{\partial g}{\partial s}(20, 15) = -0.5 \).

(i) (13 points) Find a linear approximation to \( g(p, s) \) around \((p, s) = (20, 15)\).

**Solution:**

\[
L(p, s) = g(20, 15) + \left[ \frac{\partial g}{\partial p}(20, 15) \right] (p - 20) + \left[ \frac{\partial g}{\partial s}(20, 15) \right] (s - 15)
\]

\[
= 2 - 0.5(p - 20) + 0.3(s - 15).
\]

I assigned 10 points for knowing the formula, and 3 points for the rest.

(ii) (7 points) Use the linear approximation to predict your grade should you party 21 hours and study 17 hours.

**Solution:**

\[
L(21, 17) = 2 - 0.5(21 - 20) + 0.3(2) = 2.1.
\]

I assigned 7 points for calculation. If the only mistake one made in part (i) was switching 0.3 and -0.5 around, I did not penalize them for getting 1.3 for an answer.
4. (30 points) Let $h(u, v)$ be a differentiable function with
\[
\frac{\partial h}{\partial u}(9, 7) = 2
\]
and
\[
\frac{\partial h}{\partial v}(9, 7) = -1.
\]

Let $z(x, y) = h\left(f(x, y), g(x, y)\right)$ where $u = f(x, y) = x^2 - y^2$ and $v = g(x, y) = 2xy - 33$. Find $\frac{\partial z}{\partial x}(5, 4)$ and $\frac{\partial z}{\partial y}(5, 4)$.

**Solution:** Chain rule gives
\[
\frac{\partial z}{\partial x} = \frac{\partial h}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial h}{\partial v} \frac{\partial v}{\partial x}
\]
where
\[
u = f(x, y) = x^2 - y^2 	ext{ and } v = g(x, y) = 2xy - 33
\]

At $x = 5$ and $y = 4$, we have $u = 9$ and $v = 7$.

Therefore,
\[
\frac{\partial z}{\partial x}(5, 4) = \frac{\partial h}{\partial u}(9, 7) \frac{\partial u}{\partial x}(5, 4) + \frac{\partial h}{\partial v}(9, 7) \frac{\partial v}{\partial x}(5, 4)
\]
\[
\frac{\partial h}{\partial u}(9, 7) = 2 \text{ and } \frac{\partial h}{\partial v}(9, 7) = -1 \text{ are given}
\]
\[
\frac{\partial u}{\partial x}(x, y) = 2x \text{ and } \frac{\partial v}{\partial x}(x, y) = 2y
\]
which gives $\frac{\partial u}{\partial x}(5, 4) = 10$ and $\frac{\partial v}{\partial x}(5, 4) = 8$.

Finally
\[
\frac{\partial z}{\partial x}(5, 4) = (2)(10) + (-1)(8) = 12
\]

Similarly
\[
\frac{\partial z}{\partial y}(5, 4) = \frac{\partial h}{\partial u}(9, 7) \frac{\partial u}{\partial y}(5, 4) + \frac{\partial h}{\partial v}(9, 7) \frac{\partial v}{\partial y}(5, 4)
\]
which gives
\[
\frac{\partial z}{\partial y}(5, 4) = -26
\]
5. (15 points) Where does the tangent plane at \((x, y, z) = (1, 2, -5)\) to the graph \(z = f(x, y)\) of \(f(x, y) = \frac{2x^2}{y} - 3xy\) meet the \(z\)-axis? (Your answer will be a point of the form \((0, 0, a)\) or \(a\vec{k}\).)

**Solution:** Tangent plane to the graph of the function \(f(x, y)\) at the point \((a, b)\) (if it exists) is of the form

\[
z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)
\]

\[
\frac{\partial f}{\partial x}(x, y) = \frac{4x}{y} - 3y
\]

and

This implies

\[
\frac{\partial f}{\partial x}(1, 2) = -4
\]

\[
\frac{\partial f}{\partial x}(x, y) = -\frac{2x^2}{y^2} - 3x
\]

which gives \(\frac{\partial f}{\partial x}(1, 2) = -\frac{7}{2}\)

Therefore, the tangent plane

\[
z = -4x - \frac{7}{2}y + 6
\]

Tangent plane meets \(Z\)-axis when \(x = 0\) and \(y = 0\), hence at \((0, 0, 6)\) or \(6k\).
6. (25 points) Consider the hyperboloid given by \( g(x, y, z) = 2x^2 + 3y^2 - z^2 = 7 \).

(i) (15 points) Find the tangent plane to the hyperboloid at the point \((x, y, z) = (2, 1, -2)\).

**Solution:** The answer is the plane

\[
8x + 6y + 4z - 14 = 0.
\]

Here, we use the fact that the tangent plane at a point \((x_0, y_0, z_0)\) to a surface (defined as a level curve such as \(g(x, y, z) = c\)) is given by

\[
\nabla g(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.
\]

We know that \((x_0, y_0, z_0) = (2, 1, -2)\), so we merely need to find the gradient.

We have in general that

\[
\nabla g(x, y, z) = \langle \partial g/\partial x, \partial g/\partial y, \partial g/\partial z \rangle,
\]

so in our case this is

\[
\nabla g(x, y, z) = \langle 4x, 6y, -2z \rangle.
\]

So \(\nabla g(2, 1, -2) = \langle 8, 6, 4 \rangle\).

So the equation for our plane is

\[
\langle 8, 6, 4 \rangle \cdot \langle x - 2, y - 1, z + 2 \rangle = 0,
\]

which is

\[
8x + 6y + 4z - 14 = 0.
\]

Grading Scheme:
We gave 8 points for the correct determination of the normal vector (i.e. finding the gradient as above), and 7 points for using a valid equation for the plane.

(ii) (10 points) Can the tangent plane to the hyperboloid ever be horizontal? If so, find a point on the hyperboloid where the tangent plane is horizontal. If not, explain why not.

**Solution:** The answer is no, there is no point on the hyperboloid that has a horizontal tangent plane.

Having a horizontal tangent plane would mean that in the equation for the tangent plane, \(Ax + By + Cz + D = 0\), \(A\) and \(B\) are zero and \(C\) is not.

Thus, this means that for

\[
\langle A, B, C \rangle = \nabla g(x_0, y_0, z_0) = \langle 4x_0, 6y_0, -2z_0 \rangle,
\]

we need \(x_0 = y_0 = 0\). But there is no such point on the hyperboloid: we would need to find \(z_0\) so that \(2 \cdot 0 + 3 \cdot 0 - z_0^2 = 7\), or that \(z_0 = \sqrt{-7}\), which is impossible.

Grading Scheme:
Aside from correct answers as above, I gave credit for knowing what horizontal would mean for a tangent plane (1 or 2 points), and for answers that, while incorrect, would have lead to the correct answer eventually.