Today: 2.3 Differentiation

What does it mean for a function of several variables to be differentiable?

For $f : \mathbb{R}^2 \to \mathbb{R}$, what does it mean for $f(x, y)$ to have a derivative at the point $(x_0, y_0)$?

Intuitively, we would like there to be a well-defined plane that is tangent to the graph of $f$ at $(x_0, y_0)$. In other words, the graph of $f$ should have no sharp corners or folds at that point.

On the way to understanding what that means precisely, we introduce the notion of partial derivative.

Partial derivatives: definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function of $n$ variables. Write $z = f(x_1, x_2, \ldots, x_n)$.

The $j$th partial derivative of $f$, for $j = 1, 2, \ldots, n$ is the function $\frac{\partial f}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\frac{\partial f}{\partial x_j}(x_1, x_2, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_j + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}$$

Partial derivatives: visual intuition

Last time, we discussed level sets and sections as tools to understand graphs of functions of several variables.

For $f : \mathbb{R}^2 \to \mathbb{R}$, a section of its graph is the intersection with a vertical plane, usually with either $x$ or $y$ fixed. That intersection is a curve in a plane.

Think of partial derivatives as finding slopes of tangent lines (via one-variable derivatives) to those sections in the vertical planes.

We can write that definition in vector notation:

$$\frac{\partial f}{\partial x_j}(x) = \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h}$$

Usually we work with functions of two variables $z = f(x, y)$, in which case we can write

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

and

$$\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$
Example 1. a. If \( f(x, y) = x \cos(xy) \), find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \).
b. Then, evaluate those functions at \((0, 0)\) and \((\sqrt{2}, -\sqrt{2})\).
c. Then, use this information to find tangent planes to the graph of \( f \) at \((0, 0)\) and \((\sqrt{2}, -\sqrt{2})\).

Provisionally, let us say that the tangent plane to the graph of \( f(x, y) \) at \((x_0, y_0)\) is given by the equation
\[
z = f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0).
\]

That type of example is why we said “Provisionally” before, and that is why partial derivatives on their own are not enough to define differentiability for a function of several variables.

We don’t want to call a function differentiable unless that plane we defined is a good approximation to the graph of the function, as follows:

The function \( f(x, y) \) is differentiable at \((x_0, y_0)\) if the partial derivatives exist at that point and if
\[
\lim_{(x, y) \to (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)}{||(x, y) - (x_0, y_0)||} = 0
\]
as \((x, y) \to (x_0, y_0)\).

Example 2. a. If \( f(x, y) = x^\frac{1}{2} - y^\frac{1}{2} \), find \( \frac{\partial f}{\partial x}(0, 0) \) and \( \frac{\partial f}{\partial y}(0, 0) \).
b. What would that information say about the tangent plane to the graph of \( f \) at \((0, 0)\)?

On Friday, we will want to extend this notion of differentiability to functions \( f : \mathbb{R}^n \to \mathbb{R}^m \).

To do that, we will consider the \( m \times n \) matrix of partial derivatives \( Df \):
\[
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix}
\]

Example 3. Let \( f(x, y, z) = (e^{yz}, e^{xz}, e^{xy}) \). Find \( Df \).