MATH 2374.020 – Lecture 8 – 2/15/08

Place HW 03 in folders in back center of the room.

HW 04: due Fri. 2/22
2.4 #2, 16, 18
2.5 #2e, 5d, 6d, 11
2.6 #3a, 4b, 7b, 13a, 16
Sample final #7, 9a

Quiz 09: due Wed. 2/20, 9 a.m.
Quiz 10: due Fri. 2/22, 9 a.m.
Exam 1: 2/27. Will cover chapters 1 and 2. That is, through the material that we cover today and that you hand in HW for next Friday.

Today:

1. Review chain rule
2. Gradient and directional derivative

Example 4 from Wed. Let \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) be given by \( f(x, y, z) = (-y, x, -z) \). Let \( c(t) = (1, t, \sin t) \).

Find \( D(f \circ c)(0) \) using the chain rule, and interpret geometrically.

Notes

• I will occasionally use the notation \( D_v f(x) \) for this quantity.
• This is defined for any vector \( \mathbf{v} \), but we usually take \( \mathbf{v} \) to be a unit vector. Why?
• If \( f \) is differentiable, then all directional derivatives exist, and
  \[ D_v f(x) = \nabla f(x) \cdot \mathbf{v}. \]
• For \( f : \mathbb{R}^2 \to \mathbb{R} \), think of intersecting the graph with a vertical plane through \( x \) and parallel to \( \mathbf{v} \).

Directional derivatives

When we calculate the partial derivative of a function \( f : \mathbb{R}^3 \to \mathbb{R} \), we are asking for the rate of change of the function as the input moves in the \( i \) direction; that is, in the direction of the positive \( x \)-axis.

We can investigate the rate of change of a function as the input moves in any direction.

Definition. For \( f : \mathbb{R}^n \to \mathbb{R} \), the directional derivative of \( f \) at \( x \) in the direction of \( \mathbf{v} \) is

\[
\frac{d}{dt} f(x + tv),
\]
evaluated at \( t = 0 \), if the limit exists.
**Example 1.** Let \( f(x, y) = \sin x + \sin y \). Find the directional derivative of \( f \) at the point:

1. \( \left( \frac{\pi}{2}, -\frac{\pi}{2} \right) \) in the direction of \( v = \frac{1}{\sqrt{2}}(1, 1) \).
2. \( (0, 0) \) in the direction of \( v = (\cos \theta, \sin \theta) \) for \( 0 \leq \theta \leq 2\pi \).

**Theorem.** Let \( f : \mathbb{R}^3 \to \mathbb{R} \) have continuous partial derivatives, and suppose \((x_0, y_0, z_0)\) lies on the level surface defined by \( f(x, y, z) = k \), for some constant \( k \). Then \( \nabla f(x_0, y_0, z_0) \) is normal to the level surface.

What do we mean by normal to a surface?

We can then use that normal vector to find a plane that is tangent to the level surface at \((x_0, y_0, z_0)\).

**Example 2.** Let \( f(x, y, z) = x^2 + 2y^2 + 3z^2 \). Find the normal vector to the level surface \( f(x, y, z) = 6 \) at the point \((1, -1, 1)\), and write an equation for the tangent plane to the surface there.

The same thing works for functions of two variables, except now the gradient is normal to level curves.

**Example 3.** Let \( f(x, y) = e^{-(x^2 + 3y^2)} \).

1. Find \( \nabla f \) at the points of the level curve \( f(x, y) = 1/e \) where \( x = 1/2 \).
2. Find a vector that is normal to the graph of \( f(x, y) = e^{-(x^2 + 3y^2)} \) at \((1/2, 1/4)\), and use it to produce a plane that is tangent to the graph there. (This time, think of the graph as a level surface of a function of three variables, then notice that this produces the same plane as our previous method.)