Today:

1. (2.6) The gradient is normal to level sets.
2. (3.1) Mixed partials are equal.
3. (5.1, 5.2) The double integral

Last time, we showed that the gradient $\nabla f$ points in the direction of fastest increase of $f$.

**Theorem.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ have continuous partial derivatives, and suppose $(x_0, y_0, z_0)$ lies on the level surface defined by $f(x, y, z) = k$, for some constant $k$. Then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface.

What do we mean by normal to a surface?

We can then use that normal vector to find a plane that is tangent to the level surface at $(x_0, y_0, z_0)$.

**Example 2 from Friday.** Let $f(x, y, z) = x^2 + 2y^2 + 3z^2$. Find the normal vector to the level surface $f(x, y, z) = 6$ at the point $(1, -1, 1)$, and write an equation for the tangent plane to the surface there.

The same thing works for functions of two variables, except now the gradient is normal to level curves.

**Example 3 from Friday.** Let $f(x, y) = e^{-(x^2 + 3y^2)}$.

1. Find $\nabla f$ at the points of the level curve $f = 1/e$ where $x = 1/2$.

2. Find a vector that is normal to the graph of $f(x, y) = e^{-(x^2 + 3y^2)}$ at $(1/2, 1/2)$, and use it to produce a plane that is tangent to the graph there. (This time, think of the graph as a level surface of a function of three variables, then notice that this produces the same plane as our previous method.)

Second partial derivatives

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function of two variables, then so are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and hence we can investigate their partial derivatives with respect to $x$ and $y$:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$$

**Example 1.**

Let $f(x, y) = x^2 \sin y$. Find the four second partial derivatives.

**Theorem.** Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $f$ has continuous second partial derivatives, then the mixed second partial derivatives are equal.

That is, $f_{xy} = f_{yx}$ or $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. 
Double integrals as volume

Recall our definition of integral for functions $f : \mathbb{R} \to \mathbb{R}$.

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined on a rectangle $R = [a, b] \times [c, d]$ defined by $a \leq x \leq b$ and $c \leq y \leq d$.

Create subrectangles as follows: for each $n$, write $\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{n}$. Then there are $n^2$ subrectangles $R_{ij}$.

In each subrectangle $R_{ij}$ choose a point $(x_{ij}, y_{ij})$.

We define
\[
\int_{R} f(x, y) \, dA = \lim_{n \to \infty} \sum_{i,j=1}^{n} f(x_{ij}, y_{ij}) \Delta x \Delta y,
\]
if the limit exists.

How will we calculate these quantities?

**Example 2.** Refer to the diagram on the board. Find $\int_{R} f \, dA$.

A “slicing” principle leads to the following iterated integrals:
\[
\int_{R} f(x, y) \, dA = \int_{a}^{b} \left[ \int_{c}^{d} f(x, y) \, dy \right] dx = \int_{c}^{d} \left[ \int_{a}^{b} f(x, y) \, dx \right] dy.
\]

**Example 3.** Find
\[
\int_{R} xye^{-x^2+y^2} \, dA,
\]
where $R = [0, A] \times [0, A]$ for fixed $A > 0$.

How does this quantity behave as $A \to \infty$?