Today:

Triple integrals

Applications of triple integrals:

- mass
- average value

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First, we define the triple integral over a rectangular box:

Suppose that \( f : R^3 \to R \) is a function defined on a rectangular box \( R: a \leq x \leq b, c \leq y \leq d, e \leq z \leq f \).

For each \( n \), form \( n^3 \) sub-boxes \( R_{ijk} \) with dimensions \( \Delta x = \frac{b-a}{n} \), \( \Delta y = \frac{d-c}{n} \), \( \Delta z = \frac{f-e}{n} \), and in each sub-box, choose a point \( P_{ijk} \).

Form the Riemann sum

\[
S_n = \sum_{i,j,k=1}^{n} f(P_{ijk}) \Delta x \Delta y \Delta z.
\]

Then we define

\[
\iiint_R f(x, y, z) \, dV = \lim_{n \to \infty} S_n,
\]

if the limit exists.

Again, we can use iterated integrals (Fubini’s Theorem) to calculate:

\[
\iiint_R f(x, y, z) \, dV = \int_a^b \int_c^d \int_e^f f(x, y, z) \, dz \, dy \, dx.
\]

The three integrals can be arranged in any of six orders.

**Example 1.** Find

\[
\iiint_R f(x, y, z) \, dV
\]

where \( f(x, y, z) = xyz \) and \( R = [0, 1] \times [2, 4] \times [0, 1] \).

**Example 2.** Find \( \iiint_R f(x, y, z) \, dV \) where \( R \) is the region enclosed by the coordinate planes and the plane \( x + y + z = 1 \).

Triple integrals over “elementary regions”:

A region \( R \) is elementary if it can be described in the form

\[
f_1(x, y) \leq z \leq f_2(x, y),
\]

\[
g_1(x) \leq y \leq g_2(x),
\]

\[
a \leq x \leq b.
\]

Then

\[
\iiint_R f(x, y, z) \, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} f(x, y, z) \, dz \, dy \, dx.
\]

Similarly for the other orderings of the three variables.
Example 3. Find $\iiint_R z \, dV$ where $R$ is the region in the first octant (that is, where $x, y, z \geq 0$) enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$.

...integrate $\sin z$ over the same region?

...replace $y = 3x$ with $y = x^2$?