Review: Divergence and curl

Given a vector field $F$,

$$\text{div } F = \nabla \cdot F$$

is a real-valued function, and

$$\text{curl } F = \nabla \times F$$

is a vector field.

We saw at the end of last time that

$$\text{div } \text{curl } F = 0.$$
Line integrals of vector fields

Motivation: (Referring to diagram) What is the work done by force $F$ to move a particle from point $A$ to point $B$?

Along a path, the direction may be changing, so we divide the path into small subpaths on which the direction and the force are approximately constant.

**Definition.** The line integral of a vector field $F$ along a path $c$ on the interval $a \leq t \leq b$ is defined to be

$$\int_c F \cdot ds = \int_a^b F(c(t)) \cdot c'(t) \, dt.$$ 

Example 3. Let $c(t) = (\cos t, \sin t, t)$ for $0 \leq t \leq 2\pi$ (as in example 2). Let $F = (-x, -y, -z)$. Find

$$\int_c F \cdot ds.$$ 

Then find the value of the integral on the interval $[-2\pi, 0]$.

Example 4. Find

$$\int c (yz \, dx + xz \, dy + xy \, dz),$$

where $c(t) = (\cos t, \sin t, \cos t \sin t)$ for $0 \leq t \leq 2\pi$.

If we write $F$ in terms of its components:

$$F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)),$$

and we write $ds$ as the formal expression (a “differential form”)

$$ds = (dx, dy, dz),$$

then

$$F \cdot ds = P \, dx + Q \, dy + R \, dz,$$

and we can write

$$\int_c F \cdot ds = \int_a^b F(c(t)) \cdot c'(t) \, dt$$

$$= \int_c P \, dx + Q \, dy + R \, dz = \int_a^b (P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt}) \, dt.$$

Line integrals of gradient vector fields

Recall that we say a vector field $F$ is conservative if it is the gradient of some function $f$:

$$\nabla f = F.$$

Remember that not all vector fields are conservative.

**Theorem.** If $c$ is a path defined on $a \leq t \leq b$, and $f$ is a real-valued function, then

$$\int_c \nabla f \cdot ds = f(c(b)) - f(c(a)).$$

Back to Example 4. Do you recognize $F = (yz, xz, xy)$ as the gradient of some function?

One special kind of reparametrization

Given a path $c$ defined on the interval $a \leq t \leq b$, define

$$\bar{c}(t) = c(a + b - t),$$

the opposite path to $c$.

How do path integrals and line integrals on $c$ and $\bar{c}$ compare?
Theorem. If $F$ is vector field, then
$$\int_\overline{c} F \cdot ds = - \int_c F \cdot ds.$$ 
If $f$ is a real-valued function, then
$$\int_\overline{c} f \, ds = \int_c f \, ds.$$ 

Notice the negative sign in the first equation and the lack of one in the second!

Line integrals of vector fields respect orientation, and path integrals of real-valued functions ignore orientation.