Example 4 from Wednesday. Find
\[ \int_C yz\,dx + xz\,dy + xy\,dz, \]
where \( c(t) = (\cos t, \sin t, \cos t \sin t) \) for \( 0 \leq t \leq 2\pi \).

Recall that we calculated the value of this line integral directly (leaving out some steps) to be 0.

**Theorem.** If \( c \) is a path defined on \( a \leq t \leq b \), and \( f \) is a real-valued function, then
\[ \int_C \nabla f \cdot ds = f(c(b)) - f(c(a)). \]

One special kind of reparametrization

**Question:** How does the value of a line integral change if we travel along the path in the opposite direction?

Given a path \( c \) defined on the interval \( a \leq t \leq b \), define \( \tilde{c}(t) = c(a + b - t) \), the opposite path to \( c \).

Compare \( \int_c \mathbf{F} \cdot ds \) and \( \int_{\tilde{c}} \mathbf{F} \cdot ds \).

How do path integrals compare on \( c \) and \( \tilde{c} \)?

**Theorem.** If \( \mathbf{F} \) is vector field, then
\[ \int_c \mathbf{F} \cdot ds = -\int_{\tilde{c}} \mathbf{F} \cdot ds. \]

If \( f \) is a real-valued function, then
\[ \int_c f\,ds = \int_{\tilde{c}} f\,ds. \]

Notice the negative sign in the first equation and the lack of one in the second!

Line integrals of vector fields respect orientation, and path integrals of real-valued functions ignore orientation.

**Extremely vague Fundamental Theorem**

The integral of a “derivative” of a function on a region is equal to the total change of the function on the boundary.
**Green’s Theorem.** Let \( D \) be a simple region and let \( C \) be its boundary. Let \( C^+ \) be the path that traces out \( C \) with positive orientation. Suppose \( P \) and \( Q \) are functions defined on \( D \) that have continuous partial derivatives. Then

\[
\int_{C^+} P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.
\]

**Vector form:**

Let \( F \) be the vector field \( F = (P, Q) \). Then we can write the conclusion of Green’s theorem this way:

\[
\int_{C^+} F \cdot ds = \iint_D (\text{curl} \, F) \cdot k \, dA.
\]

Informally: the line integral of a vector field on the “outside” is equal to the double integral of the scalar curl on the “inside”.

Before we see why Green’s Theorem might be true, some applications:

We can use Green’s Theorem to find the area of the region \( D \).

\[
A(D) = \frac{1}{2} \int_{C^+} x \, dy - y \, dx.
\]

**Example 1.** Use this formula to find the area of the region enclosed by the path \((r \cos t, r \sin t)\).

**Example 2.** Find the area of the flower described in polar coordinates \((r, t)\) by the equation \( r = \sin kt \), for each \( k \geq 1 \).