Today:

Let’s get back up to speed and then return to Green’s Theorem.

A proof of Green’s Theorem in a special case.

End of material for Exam 2.

8.3. When is a vector field conservative?

Review:

We are working with vector fields, which are vector-valued functions \( F \) that assign to each point of \( \mathbb{R}^3 \) (or \( \mathbb{R}^2 \)) a vector in \( \mathbb{R}^3 \) (or \( \mathbb{R}^2 \)).

The line integral of \( F \) along a path \( c(t) \) for \( a \leq t \leq b \) is given by:

\[
\int_c F \cdot ds = \int_a^b F(c(t)) \cdot c'(t) \, dt.
\]

If \( F \) happens to be the gradient of a real-valued function \( f \) (that is \( \nabla f = F \)), then the line integral can be calculated quickly via the fundamental theorem for gradients.

**Theorem.** If \( c \) is a path defined on \( a \leq t \leq b \), and \( f \) is a real-valued function, then

\[
\int_c \nabla f \cdot ds = f(c(b)) - f(c(a)).
\]

Now we want to relate the line integral along a closed path to a double integral on the region the path encloses.

**Green’s Theorem.** Let \( D \) be a simple region and let \( C \) be its boundary. Let \( C^+ \) be the path that traces out \( C \) with positive orientation. Suppose \( P \) and \( Q \) are functions defined on \( D \) that have continuous partial derivatives. Then

\[
\int_{C^+} P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.
\]

Last time, we used Green’s Theorem to find the area of a region:

\[
A(D) = \frac{1}{2} \int_{C^+} x \, dy - y \, dx.
\]

Note: Green’s theorem applies to more complicated regions, as in the following example:

**Example 1.** Find the area of the annulus centered at the origin that has outer radius \( R \) and inner radius \( r \).
Now, why is Green’s Theorem true?

To get a feel for why it’s true, let’s prove it in the case that the region \( D \) is the rectangle defined by \( a \leq x \leq b \) and \( c \leq y \leq d \). Let \( C^+ \) be the positively oriented path starting at \((a, c)\) around the rectangle. We show:

**Step 1.**

\[
\int_{C^+} P \, dx = -\iint_R \frac{\partial P}{\partial y} \, dx \, dy.
\]

(note negative sign)

**Step 2.**

\[
\int_{C^+} Q \, dy = \iint_R \frac{\partial Q}{\partial x} \, dx \, dy.
\]

When is a vector field conservative?

Recall that we say a vector field \( \mathbf{F} \) is conservative if it is the gradient of a real-valued function. That is, \( \nabla f = \mathbf{F} \).

This is a nice thing to know: in this case,

\[
\int_C \mathbf{F} \cdot ds = f(c(b)) - f(c(a)).
\]

In other words, the value of the integral is path-independent. It depends only on the value of the function at the endpoints of the path.

The following statement gives us tools for deciding whether a field is conservative.

(Note, however, that the analogous statement in the book concerns vector fields in \( \mathbb{R}^3 \) and requires Stokes’ Theorem (8.2). For now, we work with vector fields in \( \mathbb{R}^2 \), so that part of the following statement can be proven using Green’s Theorem.)

For a \( C^1 \) vector field \( \mathbf{F} \) on \( \mathbb{R}^2 \), the following conditions are equivalent:

1. For any simple closed curve \( C \), we have \( \int_C \mathbf{F} \cdot ds = 0 \).

2. For any two simple closed curves \( C_1 \) and \( C_2 \) with the same endpoints, we have \( \int_{C_1} \mathbf{F} \cdot ds = \int_{C_2} \mathbf{F} \cdot ds \).

3. \( \mathbf{F} \) is the gradient of some function \( f \).

4. \( \text{curl} \mathbf{F} = \nabla \times \mathbf{F} = 0 \). And for vector fields in \( \mathbb{R}^2 \), \( \text{curl} \mathbf{F} = (Q_x - P_y)k \), so this is the same as saying \( Q_x = P_y \).

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**Example 2.** Is \( \mathbf{F} = (x^2 - y^2, x^2 - y^2) \) conservative? If so, find a potential function for \( \mathbf{F} \). That is, find \( f \) so that \( \nabla f = \mathbf{F} \).

**Example 3.** Is \( \mathbf{F} = (3 + 2xy, x^2 - 3y^3) \) conservative? If so, find a potential function for \( \mathbf{F} \).

**Example 4.** (8.3 #9) Find \( \int_C \mathbf{F} \cdot ds \), where \( \mathbf{F} = (e^t \sin y, e^t \cos y, z^2) \) and \( \mathbf{c}(t) = (\sqrt{t}, \sqrt{t}, e^{t^2}) \) for \( 0 \leq t \leq 1 \).