Today:

Stokes says, "Integrate over any surface you want!"

Gauss' Theorem, our final generalization of the Fundamental Theorem of Calculus

8.2 #7 Evaluate the integral
\[ \iiint_W (\nabla \times \mathbf{F}) \cdot dV, \]
where \( W \) is the portion of the surface of the sphere defined by \( x^2 + y^2 + z^2 = 1 \) and \( x + y + z \leq 1 \), and where \( \mathbf{F} = r \times (i + j + k) \), with \( r = xi + yj + zk \).

Example 3 from Wed. Find
\[ \iint_S (\nabla \times \mathbf{F}) \cdot dS, \]
where \( S \) is the part of the ellipsoid \( x^2 + y^2 + 2z^2 = 10 \) with \( z \geq 0 \), and \( \mathbf{F} = (\sin(xy), e^x, -yz) \).

Rewriting Green's Theorem to prepare for Gauss' Theorem

Divergence form of Green's Theorem
\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \text{div} \mathbf{F} \, dA. \]

Gauss' Theorem Let \( W \) be a region in \( \mathbb{R}^3 \) that is bounded by a closed, oriented surface \( S \). Let \( \mathbf{F} \) be a smooth vector field on \( W \). Then
\[ \iiint_W \mathbf{F} \cdot dS = \iiint_W \text{div} \mathbf{F} \, dV. \]

Example 1. Verify Gauss' Theorem for the ball of radius 1 about the origin in \( \mathbb{R}^3 \) and \( \mathbf{F} = (x, y, z) \).

Example 2. For \( \mathbf{F} = (y - x, x^5, y^3, \cos(x^2 y^3)) \) and \( S \) the sphere of radius 1 about the origin, find
\[ \iint_S \mathbf{F} \cdot dS. \]

Example 3. For \( \mathbf{F} = (2x, -3y, 4z) \), find
\[ \iint_S \mathbf{F} \cdot dS, \]
where \( S \) is the surface of the parallelepiped spanned by the vectors \((1, 0, 0), (0, 1, 0), \) and \((0, 2, 3)\).