1 (20 points) Using the adjoint matrix, or otherwise, compute the inverse of the matrix
\[
\begin{bmatrix}
4 & 5 & 6 & 0 \\
2 & 0 & 2 & 0 \\
1 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] .

Make sure to explain your work and or reasoning.
2 (a) **(10 points)** Determine if the vectors $v_1 = (1, -1, 2, 3)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (1, 1, 2, 1)$, and $v_4 = (4, 1, 8, 7)$ form a basis of $\mathbb{R}^4$ or not. Explain your reasoning either way.

2 (b) **(10 points)** Determine whether or not the subset of polynomials \(\{a_0 + a_1 x + a_2 x^2 + a_3 x^3, a_i \in \mathbb{R}\},\) satisfying the condition $a_0 a_1 = a_2 + a_3$ is a subspace of $P_3$. Explain your reasoning either way.
3 (a) (10 points) Verify by computation or give an explanation why $e^{-3x}\cos(4x)$ and $e^{-3x}\sin(4x)$ are two solutions to the differential equation $y'' + 6y' + 25y = 0$.

3 (b) (10 points) Use the Wronskian or otherwise to show that $e^{3x}\cos(4x)$ and $e^{3x}\sin(4x)$ are linearly independent as differentiable functions. Explain your reasoning.
4 (20 points) Compute the general solution $y(x)$ to the differential equation $y''' - 6y'' + 10y' - 6y + 9y = 0$. Explain your work.

**Remark:** I did not get to all of the relevant examples in Lecture Monday. Thus, the following examples will be discussed in Section on Tuesday.

**Hint:** Recall the **rational root test** from algebra: if a polynomial $r^d + a_{d-1}r^{d-1} + \cdots + a_2r^2 + a_1r + a_0$ has a root $r = c$, where $c$ is an integer, then $c$ must be a factor of $a_0$.

**Related Examples:** i) Solve the general solution to $y''' + y' - 10y = 0$.

**Solution:** The differential equation $y''' + y' - 10y = 0$ has characteristic equation $r^3 + r - 10 = 0$. The only possible integer roots are $\pm 1$, $\pm 2$, $\pm 5$, and $\pm 10$. Trying a few of them, we see that $r = 2$ is a root $(2^3 + 2 - 10 = 0)$. Thus we then proceed with synthetic division, dividing $r^3 + r - 10$ by $(r - 2)$ to obtain $r^3 + r - 10 = (r - 2)(r^2 + 2r + 5)$.

We find the additional two roots via the quadratic formula: $r^3 + r - 10$ has real root $r_1 = 2$ and two complex roots $-1 \pm 2i$.

We conclude that the general solution to $y''' + y' - 10y = 0$ is

$$y(x) = c_1e^{2x} + c_2e^{-x} \cos(2x) + c_3e^{-x} \sin(2x).$$

ii) Solve the IVP $y''' + 3y'' - 10y' = 0$ with $y(0) = 7$, $y'(0) = 0$, $y''(0) = 70$.

**Solution:** We have the characteristic equation $r^3 + 3r^2 - 10r = 0$. We factor this as $r(r^2 + 3r - 10) = r(r + 5)(r - 2) = 0$ which has roots $r_1 = 0$, $r_2 = 2$, $r_3 = -5$. We thus have the general solution

$$y(x) = c_1e^{0x} + c_2e^{-5x} + c_3e^{2x} = c_1 + c_2e^{-5x} + c_3e^{2x}.$$ 

Using the initial conditions, we get the linear system

$$
\begin{align*}
  y(0) &= c_1 + c_2 + c_3 = 7 \\
  y'(0) &= -5c_2 + 2c_3 = 0 \\
  y''(0) &= 25c_2 + 4c_3 = 70
\end{align*}
$$

with solutions $c_1 = 0$, $c_2 = 2$, and $c_3 = 5$. We thus get the particular solution $y(x) = 2e^{-5x} + 5e^{2x}$ for this IVP.

iii) Compute the general solution for $9y''' - 6y'' + y' = 0$.

**Solution:** We have the characteristic equation $9r^3 - 6r^2 + r^3$, which factors as $r^3(9r^2 - 6r + 1) = r^3(3r - 1)^2$. We thus have the roots $0, 0, 0, 1/3, 1/3$ with multiplicities.

Thus the general solution is $y(x) = c_1 + c_2x + c_3x^2 + c_4e^{x/3} + c_5xe^{x/3}$. 


5 (20 points) Consider the matrix \( A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & 10 & 4 \end{bmatrix} \). Find a basis for each of the subspaces:

(a) The nullspace (solution space) \( \text{Null}(A) = \{ x \in \mathbb{R}^4 \mid Ax = 0 \} \).

(b) The row space \( \text{Row}(A) \).

(c) The column space \( \text{Col}(A) \).