1) Determine the orders of the following groups of matrices, explaining your reasoning. (5 points each)

Here, \( p \) denotes a prime, a matrix \( M \) is orthogonal over a general field if and only if \( M^t M = I \), and \( T_n(F) \) denotes the the group of \( n \)-by-\( n \) invertible upper triangular matrices with entries in the field \( F \).

a) \( T_n(F_p) \)  
b) \( SL_3(F_p) \)  
c) \( O_3(F_2) \)  
d) \( SO_3(F_3) \)  
e) \( O_2(F_7) \)

2) a) (5 points) Let \( J_n \) denote the \( n \)-by-\( n \) matrix that contains a one in every single entry. For example, \( J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) and \( J_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \). Diagonalize the matrix \( J_n \), for general \( n \), explaining your work.

b) (5 points) Let \( \rho_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \). Diagonalize the matrix \( \rho_\theta \) over the complex numbers. Your answer should be written in terms of \( \theta \). For what values of \( \theta \) is \( \rho_\theta \) diagonalizable over the reals?

3) (Jordan Canonical Form) Let \( T : V \to V \) be a linear operator on a finite dimensional complex vector space \( V \). A generalized eigenvector, with value \( \lambda \in \mathbb{C} \), is a nonzero vector \( v \in V \) such that \((T - \lambda I)^k v = 0\) for some positive integer \( k \). (If this identity holds for \( k = 1 \), then \( v \) is an eigenvector with eigenvalue \( \lambda \).) We call the set of generalized eigenvectors, together with the zero vector, a generalized eigenspace, and denote it as \( V_\lambda \).

a) (5 points) Prove that for any \( \lambda \in \mathbb{C} \), \( V_\lambda \) is a \( T \)-invariant subspace.

**Hint:** Prove that \( V_\lambda \) is a subspace first, and then show \((T - \lambda I)v \in V_\lambda \). Use this to prove \( V_\lambda \) is \( T \)-invariant.
b) (10 points) Assume that \( v \in V_\lambda \) and \((T - \lambda I)^{d-1}v \neq 0 \) but \((T - \lambda I)^d v = 0\). Show that the set
\[
W_\lambda^{(v)} = \{ w_1, w_2, \ldots, w_{d-1}, w_d \} = \{ (T - \lambda I)^{d-1}v, (T - \lambda I)^{d-2}v, \ldots, (T - \lambda I)v, v \}
\]
is linearly independent.

**Hint:** Assume otherwise and apply the operator \((T - \lambda I)\) to a non-trivial linear combination.

**Remark:** You don’t have to for this problem, but it is possible to show that for each \( \lambda \in \mathbb{C} \) which is a root of \( T \)'s characteristic polynomial, one can choose a nonzero vector \( v \in V_\lambda \) so that \( W_\lambda^{(v)} \) is a basis for, i.e. spans, \( V_\lambda \). Furthermore, one can show that \( V \) is the direct sum of its generalized eigenspaces, i.e. \( V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r} \) where \( \lambda_1 \) through \( \lambda_r \) are the roots of \( T \)'s characteristic polynomial.

c) (5 points) Show that for \( w_i \in W_\lambda^{(v)} \), \( Tw_i = \lambda w_i + w_{i-1} \), and using the remark, conclude that for every linear operator \( T : V \to V \) (here, \( V \) is a complex vector space), there exists a basis such that the matrix representing \( T \) is block-diagonal where each block is from one of the following (for some choice of \( \lambda \)):
\[
\begin{bmatrix}
\lambda \\
\lambda & 1 \\
0 & \lambda
\end{bmatrix}, \begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{bmatrix}, \begin{bmatrix}
\lambda & 1 & 0 & 0 \\
0 & \lambda & 1 & 0 \\
0 & 0 & \lambda & 1 \\
0 & 0 & 0 & \lambda
\end{bmatrix}, \text{ etc.}
\]

d) (5 points) Compute the Jordan Canonical form of
\[
\begin{bmatrix}
2 & 2 & -2 \\
2 & 1 & -1 \\
2 & 1 & -1
\end{bmatrix},
\]
explaining your work.

4) a) (5 points) Let \( G \) be the group of rotational symmetries of a cube. Describe, and determine the isomorphism type of, the group \( H \), the stabilizer subgroup of a diagonal connecting two antipodal (opposite) vertices.

**(Bonus)** (5 points) Based on (a) or otherwise, prove that \( G \cong S_4 \).

b) (10 points) Let \( GL_2(\mathbb{C}) \) act on the set \( \mathbb{C}^{2 \times 2} \), of 2-by-2 complex matrices, by conjugation. Describe the orbit and stabilizer subgroup of the matrix \( \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \) under this action.

5) Let \( G \) be a group and Aut \( G \) denote the set of its automorphisms, that is isomorphisms from \( G \) to itself.

a) (5 points) Prove that Aut \( G \) is a group with the law of composition given by composition of functions.

b) (5 points) Consider the map \( \psi : G \to \text{Aut } G \) defined by \( \psi(g) = \phi_g \). Here \( \phi_g \) denotes the conjugation map from \( G \) to \( G \) which sends \( h \in G \) to \( ghg^{-1} \). Prove that \( \psi \) is a homomorphism and determine its kernel.

c) (5 points) The image of \( \psi \) is known as Inn \( G \), the subgroup of inner automorphisms. Prove that Inn \( G \) is a normal subgroup of Aut \( G \).

d) (10 points) Describe Aut \( D_4 \) and Inn \( D_4 \), and determine their isomorphism types.