Math 8669: Combinatorial Theory

HW 3: Due Monday May 5, 2014

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Remark: Please do the following four problems. No further problems will be added to this problem set.

1) (a)-(c) Compute (using combinatorial formulas rather than a computer algebra package) the Schur expansions of

\[ p_{3,2}, \quad h_4 \cdot s_{2,2,2}, \quad \text{and} \quad s_{3,2} \cdot s_{4,1}. \]

2) (a) How many Standard Young Tableaux are there of shape \((4, 2, 1, 1)\)?

(b) Draw all SYT of shape \((2, 2, 1, 1)\).

(c) Find the number of permutations in \(S_{12}\) with a longest increasing sequence of length 6 and longest decreasing sequence of length 4.

3) In this problem, you will give an inductive proof of the **Hook Length Formula**, i.e. the number of Standard Young Tableaux, of shape \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n\), is given by

\[ f^\lambda = \frac{n}{\prod_{c \in \lambda} h_c}, \]

where \(h_c\) is the number of boxes in the hook emanating from cell \(c\).

Let \(\ell_i = \lambda_i + k - i\) (assuming that \(\lambda_1 \geq \lambda_2 \geq \ldots\))

Let \(\Delta(\ell_1, \ldots, \ell_k) = \prod_{i<j}(\ell_i - \ell_j)\).

Let \(F(\ell_1, \ldots, \ell_k) = \frac{n! \Delta(\ell_1, \ldots, \ell_k)}{\ell_1! \ell_2! \cdots \ell_k!}\).
(a) Show that \( \prod_{c \in \lambda} h_c = F(\ell_1, \ldots, \ell_k) \).

(b) Show that
\[
\sum_{i=1}^{k} x_i \Delta(x_1, \ldots, x_i + t, \ldots, x_k) = (x_1 + x_2 + \cdots + x_k + \binom{k}{2} t) \Delta(x_1, \ldots, x_k).
\]

(c) Show that \( n \cdot \Delta(\ell_1, \ldots, \ell_k) = \sum_{i=1}^{k} \ell_i \Delta(\ell_1, \ldots, \ell_i - 1, \ldots, \ell_k) \).

(d) Show that \( f^\lambda = \sum_{i=1}^{k} f^{(\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_k)} \), where we define \( f^{(\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_k)} \) to be zero if \( \lambda_i = \lambda_{i+1} \).

(e) Conclude that \( f^\lambda = \prod_{c \in \lambda} h_c \).

4) (a) Recall that the symmetry group of the cube is \( S_4 \). Let \( V \) be the representation obtained by \( S_4 \) acting on the six faces of the cube. How does \( V \) decompose into irreducibles?

(b) Consider the Spect module \( V = S^{3.2} \) (an irreducible representation of \( S_{\{1,2,3,4,5\}} \)) and the trivial representation \( W \) for \( S_{\{6,7,8\}} \cong S_3 \). What is the decomposition of \( V \otimes W \uparrow_{S_6 \times S_3}^{S_8} \) into irreducible representations?