# Math 8680: Cluster Algebras and Quiver Representations

Homework 3 (Due Wednesday April 27, 2011)

I encourage collaboration on the homework, as long as each person understands the solutions, writes them up in their own words, and indicates on the homework page their collaborators. You may use computer algebra packages for calculations but should also briefly describe your calculations in words in this case.

Please do at least three of the following four problems.

1) (Similar to a problem on HW 2, with a larger example.) Let $Q_1$ be the bipartite $A_4$ quiver shown here.

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4$$

a) By mutating at only sinks, compute all the cluster variables in the corresponding cluster algebra.

b) Organize this data into a (Conway Coxeter) Frieze pattern (a checkerboard pattern with 1's down the left and right-hand sides so that for every diamond (see below) in this pattern, we have $ad - bc = 1$.)

$$\begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array}$$

c*) Compare this calculation with the Auslander-Reiten quiver of the path algebra $kQ_1$.

d*) Let $C^+$ and $C^-$ denote the composition of reflection functors corresponding to Coxeter elements. What does the action of $C^+$ or $C^-$ do to the indecomposables corresponding to the cluster variables in the above Frieze pattern, or alternatively to the indecomposables appearing in the AR quiver?

2) (This problem illustrates differences between imaginary and real roots but highlights the role of imaginary roots in a cluster algebra of affine type.) Let $Q_2$ be the quiver of type $\widetilde{A}_2$ shown below.

a) Write down $(\alpha_i, \alpha_j)$ for $1 \leq i, j \leq 3$. Recall that $(\alpha_i, \alpha_j)$ is the $(i, j)$th entry of the corr. Cartan matrix.

b) Using this or otherwise, describe the set of real positive roots for the corresponding root system.

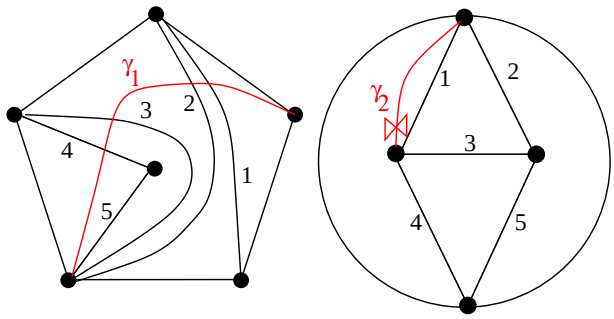
c) What are the imaginary roots of this root system?

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}$$

d*) Pick two or three indecomposable representations whose dimension vector is an imaginary root. Use the Caldero Chapoton map to obtain a Laurent polynomial corresponding to these representations. Can you see what these Laurent polynomials signify in terms of the cluster algebra?

**Hint:** Think about the surface model.

3) (This problem highlights the application of Auslander-Reiten theory to cluster algebras from surfaces.) Let $(S_1, M_1)$ and $(S_2, M_2)$ be the punctured surfaces on the left and right sides of the figure below, respectively.




a) What finite type or affine type cluster algebras do $(S_1, M_1)$ and $(S_2, M_2)$ correspond to?

b) For the finite type example, compute the corresponding AR Quiver starting with the projective indecomposables on the left.

c) Note that each slice of the AR Quiver corresponds to a cluster of the corresponding cluster algebra. Find at least two other tilting sets $\Upsilon$ that do not correspond to a slice. Let $T_\Upsilon$ denote the corresponding triangulation on the surface. What is the effect on $T_\Upsilon$ by translating the tilting set $\Upsilon$ through the AR Quiver by the Auslander-Reiten translate $\tau$?

d) For the affine type example, what arcs on the surface correspond to the *regular* indecomposable modules?

4) (This problem highlights computation of cluster variables in surfaces.)

a) and b) Let $\gamma_1$ and $\gamma_2$ be the following tagged arcs in marked surfaces $(S_1, M_1)$ and $(S_2, M_2)$ shown above. Compute the Laurent expansion (with principal coefficients) for the cluster variables $x_{\gamma_1}$ and $x_{\gamma_2}$.

c) and d) Do the same for the arcs $\gamma_3$, $\gamma_4$ in the once-punctured torus, shown here: