Lecture 21: Cluster Algebras from Surfaces
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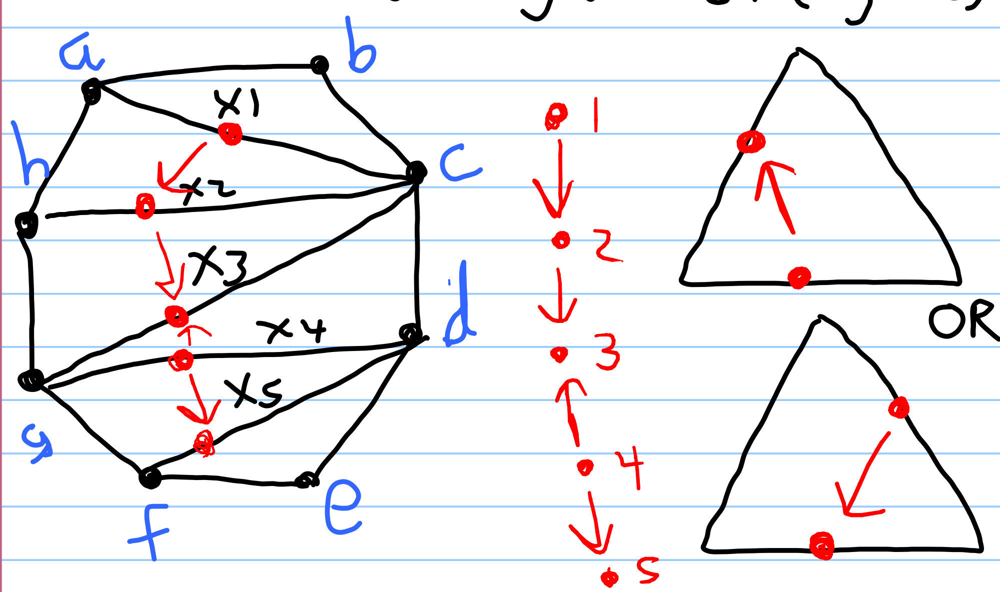
Note Title 4/5/2011

(1) We now return to discussing cluster algebras, talking about a certain class of them: based on surfaces.

<u>Fomin - Shapiro - Thurston</u> (based on cluster variety construction of Fock-Goncharov and conn. to Weil-Petersen form by Gekhtman - Shapiro - Vainshtein.

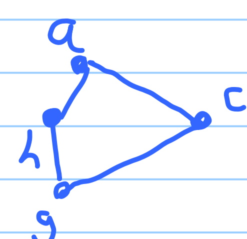
We already saw basic examples:

Cluster algebra from polygon $(n+3)$-gon
$\leftrightarrow$ Type A_n Cluster Algebra
$\leftrightarrow$ Coordinate ring of $Gr(2, n+3)$



$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & g_1 & h_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & g_2 & h_2 \end{bmatrix}$$

and for any quadilateral, e.g.



$$\underbrace{\begin{vmatrix} a_1 & g_1 \\ a_2 & g_2 \end{vmatrix} \cdot \begin{vmatrix} c_1 & h_1 \\ c_2 & h_2 \end{vmatrix}}_{X_2} = \underbrace{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \cdot \begin{vmatrix} g_1 & h_1 \\ g_2 & h_2 \end{vmatrix}}_{X_1} + \underbrace{\begin{vmatrix} a_1 & h_1 \\ a_2 & h_2 \end{vmatrix} \cdot \begin{vmatrix} c_1 & g_1 \\ c_2 & g_2 \end{vmatrix}}_{X_3}$$

$$a_1 c_1 g_2 h_2 - a_1 c_2 g_2 h_1 - a_2 c_1 g_1 h_2 + a_2 c_2 g_1 h_1$$

$$② \quad (a_1 g_2 - a_2 g_1)(c_1 h_2 - c_2 h_1) =$$

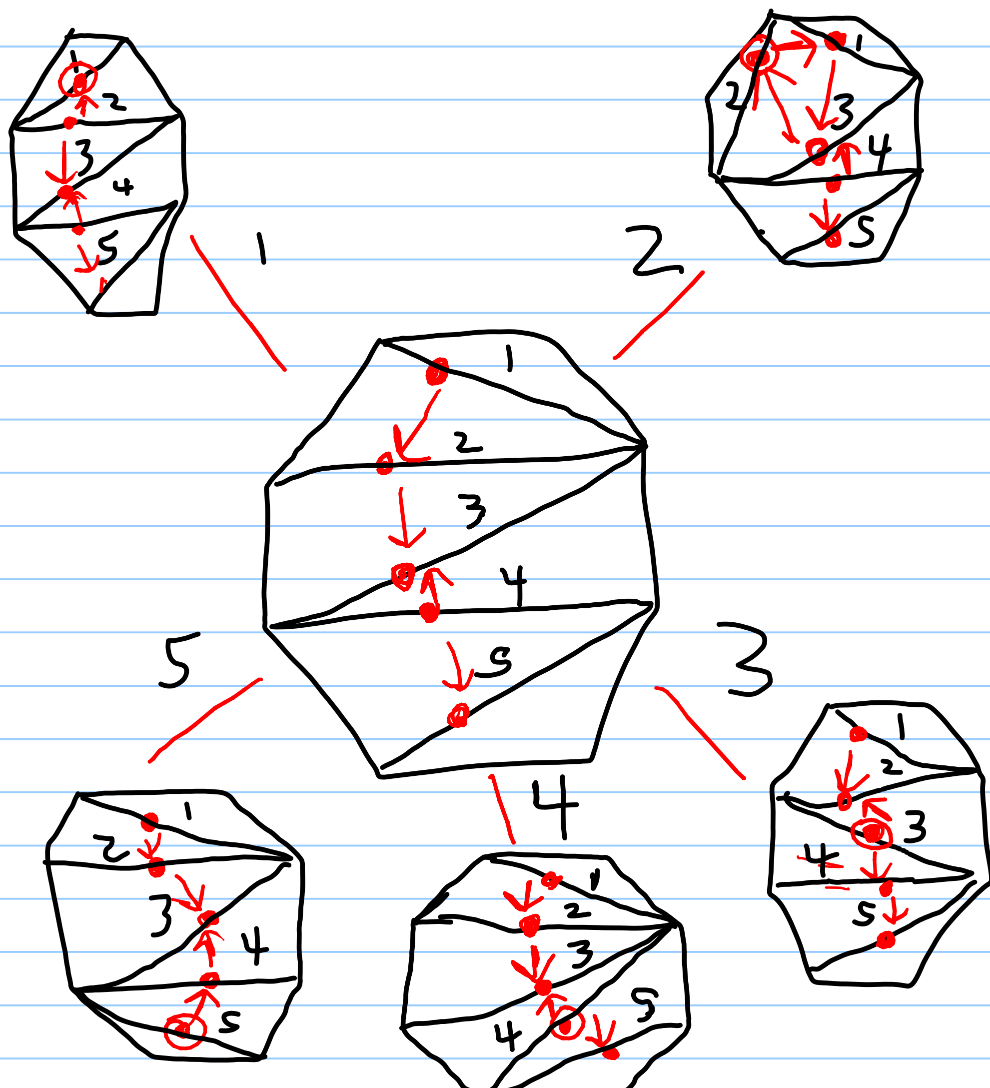
$$(a_1 c_2 - a_2 c_1)(g_1 h_2 - g_2 h_1)$$

$$+ (a_1 h_2 - a_2 h_1)(c_1 g_2 - c_2 g_1) =$$

$$\overline{a_1 c_2 g_1 h_2} - a_1 c_2 g_2 h_1 - a_2 c_1 g_1 h_2 + \overline{a_2 c_1 g_2 h_1}$$

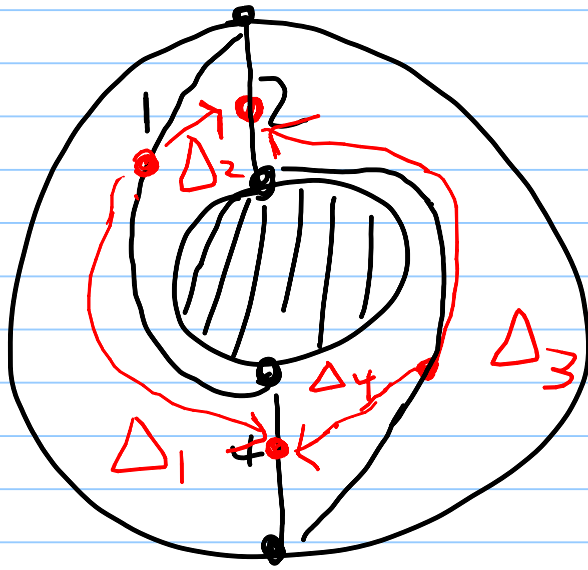
$$+ a_1 c_1 g_2 h_2 - \underline{a_1 c_2 g_1 h_2} - \underline{a_2 c_1 g_2 h_1} + a_2 c_2 g_1 h_1$$

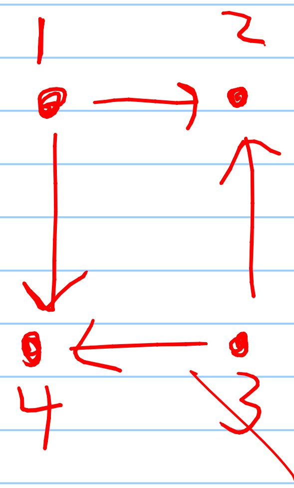
Note that if we flip a quadrilateral, equivalent to quiver mutation



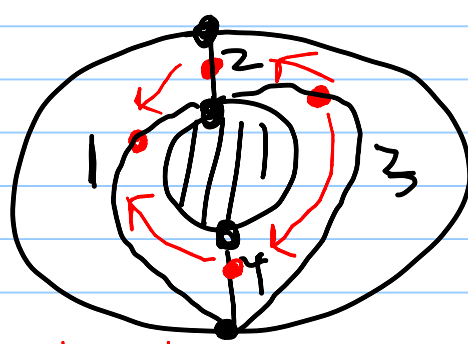
③ In fact, can always read a quiver off of a <u>triangulated surface</u>, hence yielding an exchange matrix for a cluster alg. seed.

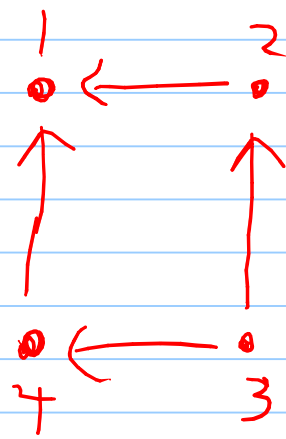
<u>Example:</u> Annulus with 4 marked points



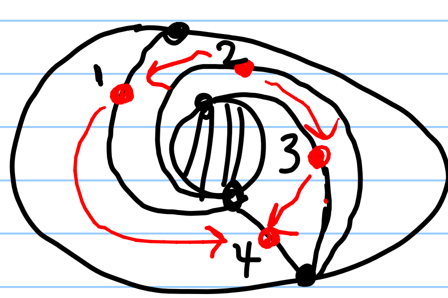


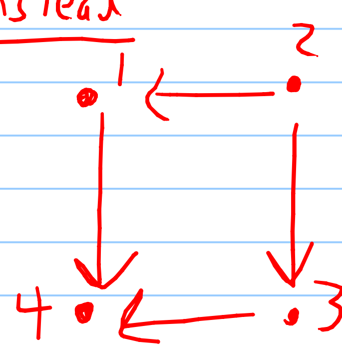
<u>Mutating/Flipping 1</u>





<u>Mutating/Flipping 2 instead</u>





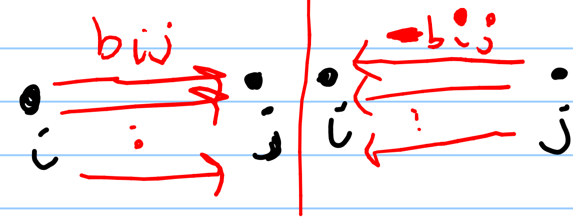
4 Thus, this original triangulation corresponds to exchange matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{\mu_1} \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\Big\downarrow \mu_2$$

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$$

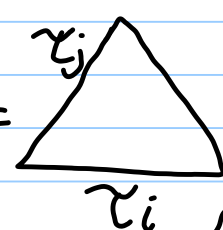
$b_{ij} > 0$	$b_{ij} < 0$

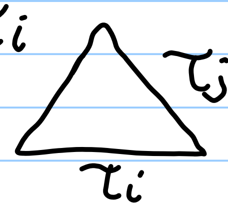


We actually can define exchange matrix B_T directly from triangulation

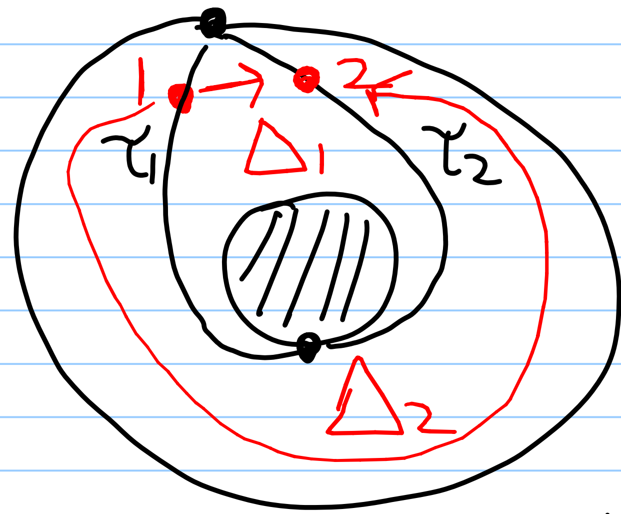
$$B_T = \sum_{\Delta \in T} B_\Delta \qquad \text{where}$$

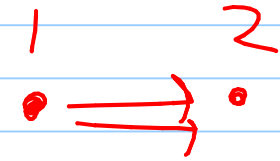
$$B_\Delta\big|_{ij} = \begin{cases} 1 & \text{if } \Delta = \\ -1 & \text{if } \Delta = \\ 0 & \text{o.w.} \end{cases}$$





5 Example (Annulus with two marked points)



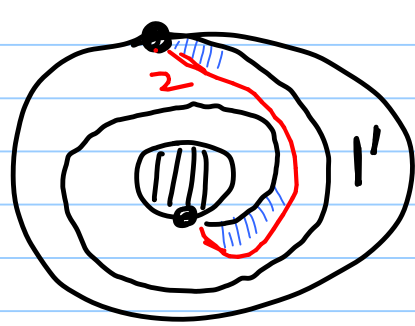


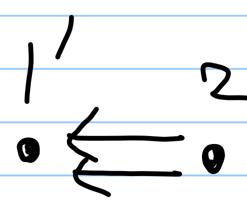
Kronecker Quiver

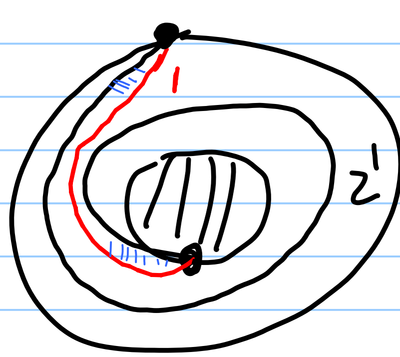
Notice that in both Δ_1 & Δ_2, τ_2, as a side of Δ_i, follows τ_1 clockwise.

$$\Rightarrow B_T = B_{\Delta_1} + B_{\Delta_2} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

Mutating/Flipping 1 or 2







(6) Even though we can get an infinite # of arcs in the annulus, if we have two arcs $\{\gamma_1, \gamma_2\}$ that don't cross each other, then their relation to one another is: γ_i follows γ_j clockwise in both triangles of the triangulation

$[i,j] = [1,2]$ or $[2,1]$.

Thm [Fomin-Shapiro-Thurston]

Let S be an orientable Riemann surface with a marking M (but w/o punctures) [M is a set of points in ∂S (the boundary)].

Then, there is a dictionary:

initial seed $\longleftrightarrow$ triangulation T (maximal collection of non-intersecting non-isotopic arcs)

initial cluster variables $\longleftrightarrow$ initial arcs of T

other cluster variables $\longleftrightarrow$ other arcs of T, i.e. a path $\gamma \in S$ whose endpoints are in M but $\operatorname{Int}\gamma \cap (M \cup \partial S) = \emptyset$.

Ptolemy relation $\updownarrow$ Cluster mutation $\longleftrightarrow$ Flipping quadrilateral

⑦ Rem: It is not apriori that all maximal collections of non-intersecting non-isotopic arcs would have the same size but in fact can be calculated as

$$n = 6g + 3c + b - 6$$

where g = genus of the surface

c = # boundary components

b = total # marked pts on all boundaries

e.g.'s (m+3)-gon

$$n = 6(0) + 3(1) + (m+3) - 6$$
$$= m$$

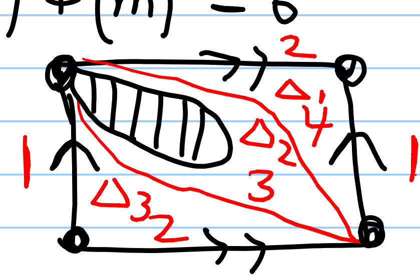
annulus with m marked points

$$n = 6(0) + 3(2) + (m) - 6$$
$$= m$$

Torus with one boundary and m marked points

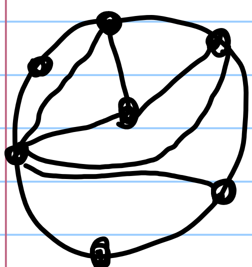
$$n = 6(1) + 3(1) + (m) - 6$$
$$= m+3$$

e.g. $m=1$ →

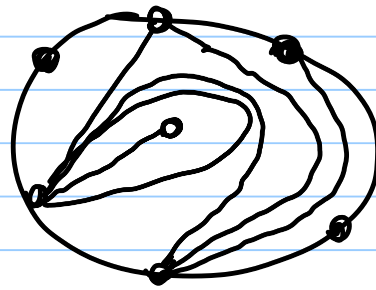


⑧ Next week we will see a generalization where punctures (i.e. $M \cap (S - \partial S) \neq \emptyset$)
Int S

such as



OR



once-punctured polygon

Let us spend the rest of today focusing on