We now return to discussing cluster algebras, talking about a certain class of them: based on surfaces.


We already saw basic examples:

Cluster algebra From polygon \((n+3)\)-gon
\leftrightarrow Type An Cluster Algebra
\leftrightarrow Coordinate ring of \(Gr(2, n+3)\)

\[
\begin{bmatrix}
  a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & g_1 & h_1 \\
  a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & g_2 & h_2
\end{bmatrix}
\]

and for any quadrilateral, e.g.,

\[
|a_1 \ g_1 \ o \ c_1 \ h_1| = |a_1 \ c_1 \ o \ g_1 \ h_1| + |a_1 \ h_1 \ o \ c_1 \ g_1|
\]
\[ a_1c_2g_2h_2 = a_1c_2g_2h_1 - a_2c_1g_1h_2 + a_2c_2g_1h_1 \]

\[ (a_1g_2 - a_2g_1)(c_1h_2 - c_2h_1) = \\
(a_1c_2 - a_2c_1)(g_1h_2 - g_2h_1) \\
+ (a_1h_2 - a_2h_1)(c_1g_2 - c_2g_1) = \\
\overline{a_1c_2g_1h_2} - \overline{a_1c_2g_2h_1} - \overline{a_2c_1g_1h_2} + \overline{a_2c_1g_2h_1} \\
+ \overline{a_1c_1g_2h_2} - \overline{a_1c_2g_1h_2} - \overline{a_2c_1g_2h_1} + \overline{a_2c_2g_1h_1} \\
\]

Note that if we flip a quadrilateral, equivalent to quiver mutation.
In fact, one can always read a quiver off of a triangulated surface, hence yielding an exchange matrix for a cluster alg. seed.

**Example:** Annulus with 4 marked points

**Mutating/Flipping 1**

**Mutating/Flipping 2 instead**
Thus, this original triangulation corresponds to exchange matrix

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[\rightarrow\]

\[
\begin{bmatrix}
0 & 1 & 0 & -1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & -1 & 0 \\
\end{bmatrix}
\]

\[M_1\]

\[
\begin{bmatrix}
0 & -1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & -1 & 0 \\
\end{bmatrix}
\]

\[M_2\]

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
\text{bij} > 0 \quad \text{bij} < 0
\]

We actually can define exchange matrix \( B_T \) directly from triangulation

\[
B_T = \sum_{\Delta \in T} B_{\Delta}
\]

where

\[
B_{\Delta}|_{ij} = \begin{cases} 
1 & \text{if } \Delta = \tau_i \\
-1 & \text{if } \Delta = \tau_i \tau_j \\
0 & \text{otherwise}
\end{cases}
\]
Example (Annulus with two marked points)

\[ \begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
\end{array} \\
\end{array} \]

Kronecker Quiver

Notice that in both \( \Delta \) and \( \Delta_2 \), \( \gamma_1 \) as a side of \( \Delta_i \) follows \( \gamma_1 \) clockwise.

\[ B_T = B_{\Delta_1} + B_{\Delta_2} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \]

Mutating/Flipping 1 or 2

\[ \begin{array}{cc}
\begin{array}{cc}
1' & 2 \\
\end{array} \\
\end{array} \]

\[ \begin{array}{cc}
\begin{array}{cc}
1 & 2' \\
\end{array} \\
\end{array} \]
Even though we can get an infinite \# of arcs in the annulus, if we have two arcs \( \gamma_1, \gamma_2 \) that don't cross each other then their relation to one another is: \( \gamma_1 \) follows \( \gamma_2 \) clockwise in both triangles of the triangulation \([i, j] = [1, 2] \) or \([2, 1]\).

**Thm [Fomin-Shapiro-Thurston]**

Let \( S \) be an orientable Riemann surface with a marking \( M \) (but w/o punctures) \([M \) is a set of points in \( \partial S \) the boundary].

Then, there is a dictionary:

- initial seed \( \leftrightarrow \) triangulation \( T \)
  - maximal collection of non-intersecting non-isotopic arcs

- initial cluster \( \leftrightarrow \) initial arcs of \( T \)

- other cluster \( \leftrightarrow \) other arcs of \( T \)
  - i.e. a path \( \gamma \in S \) whose endpoints are in \( M \) but \( \text{Int} \gamma \cap (M \cup \partial S) = \emptyset \).

- Ptolemy relation \( \leftrightarrow \) flipping quadrilateral
Rem: It is not a priori that all maximal collections of non-intersecting non-isotopic arcs would have the same size, but in fact can be calculated as

\[ n = 6g + 3c + b - 6 \]

where \( g \) = genus of the surface

\( c \) = \# boundary components

\( b \) = total \# marked pts on all boundaries

\textbf{e.g.'s}

\underline{(m+3)-gon}

\[ n = 6(0) + 3(1) + (m+3) - 6 = m \]

\underline{annulus with \( m \) marked points}

\[ n = 6(0) + 3(2) + (m) - 6 = m \]

\underline{Torus with one boundary and \( m \) marked points}

\[ n = 6(1) + 3(1) + (m) - 6 = m + 3 \]

\textbf{e.g.} \( m = 1 \)
Next week we will see a generalization where punctures (i.e., $M \cap (S-\Delta) \neq \emptyset$) such as

\begin{align*}
\text{Ints} & \quad \text{OR} \\
\text{once-punctured polygon}
\end{align*}

Let us spend the rest of today focusing on $\bullet \Rightarrow \bullet$.

\begin{align*}
X_1 & \quad X_2 & \quad M_1 \\
X_3 & \quad M_2 \\
X_4 & \quad M_1 \\
X_5
\end{align*}