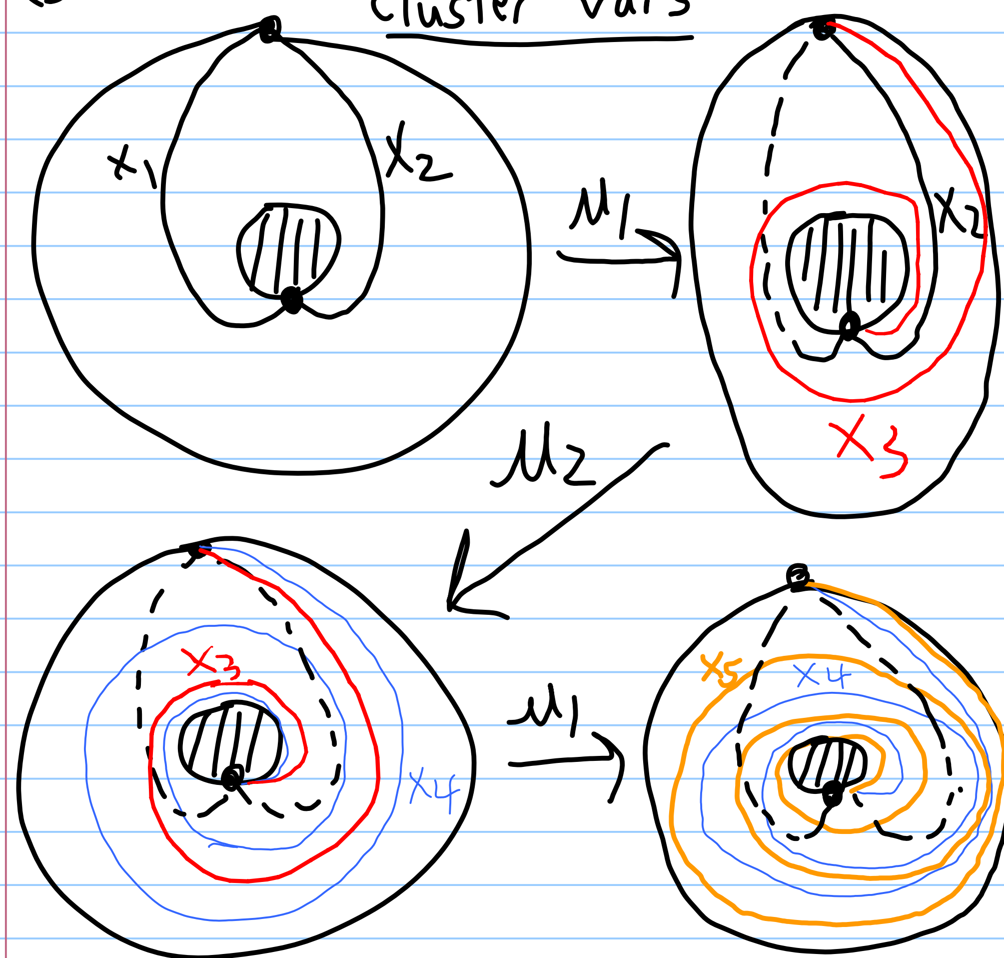
Lecture 22: Cluster Algebra from surfaces II   Gregg Musiker 8680 4-11-11

Note Title 4/10/2011

① More on the annulus e.g.

$Q \quad \underset{1}{\bullet} \Rightarrow \underset{2}{\bullet} \qquad B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$(S,M) \quad \underbrace{\{x_n x_{n-2} = x_{n-1}^2 + 1 : n \in \mathbb{Z}\}}_{\text{cluster vars}}$



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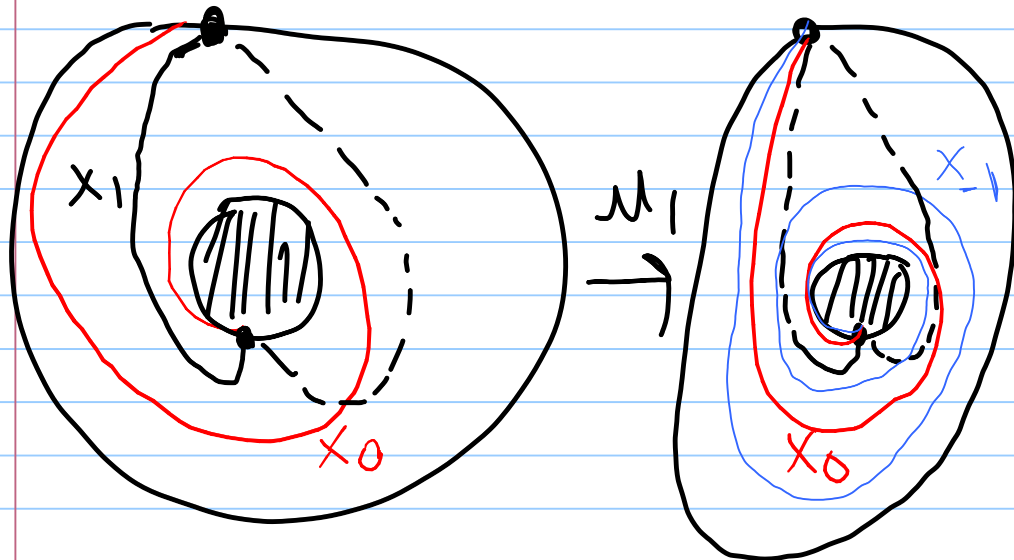
Let τ_n denote the arc corresp. to cluster variable x_n.

τ_3 crosses τ_1 once, τ_2 zero times

τ_4 crosses τ_1 twice, τ_2 once, ...

τ_5 " " τ_1 three times, τ_2 twice.

② If we mutate instead by μ_2 first:



$\ldots$ τ_0 crosses τ_2 once, τ_1 zero times

τ_{-1} " " τ_2 twice, τ_1 once.

So crossing behavior summarizes as

$$x_n = \underbrace{\mu_2\mu_1\cdots\mu_2\mu_1}_{(n-2)\text{ mutations}} x_1$$

if n is even, $n \geq 3$;

$$= \underbrace{\mu_1\cdots\mu_2\mu_1}_{(n-2)\text{ mutations}} x_1$$

if n is odd, $n \geq 3$;

$$= \underbrace{\mu_1\mu_2\cdots\mu_1\mu_2}_{(n-2)\text{ mutations}} x_2$$

if n is even, $n \leq 0$;

$$= \mu_2\cdots\mu_1\mu_2 x_2$$

if n is odd, $n \leq 0$.

③ $\{x_3, x_4, x_5, \dots\} \cup \{x_0, x_{-1}, x_{-2}, \dots\}$
$= \{\text{cluster variables}\}$

if $n \geq 3$,

τ_n crosses τ_1 $(n-2)$ times,
τ_2 $(n-3)$ times

if $n \leq 0$,

" " τ_1 $(-n-1)$ times,
τ_2 $(-n)$ times.

In this case,

$$\mathrm{denom}(x_n) = \begin{cases} x_1^{n-2} x_2^{n-3} & \text{if } n \geq 3 \\ x_1^{-n-1} x_2^{-n} & \text{if } n \leq 0 \end{cases}$$

agreeing with crossing numbers.

Recall that

$\{(k+1)\alpha_1 + k\alpha_2\} \cup \{k\alpha_1 + (k+1)\alpha_2\}$
$= \boxed{\text{Pos real roots}}$ for $\underset{1}{\bullet} \Rightarrow \underset{2}{\bullet}$

Warnings 1) For general surfaces, formula for

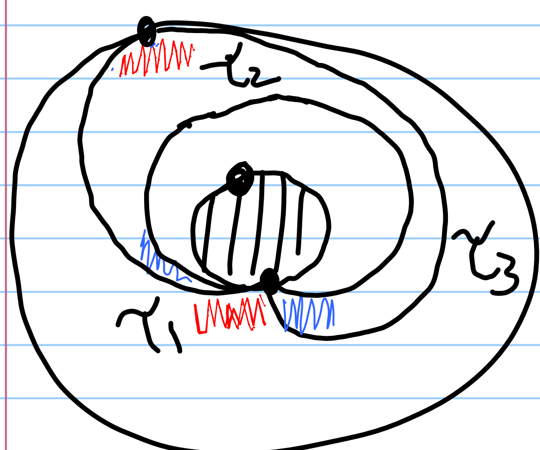
denom(cl. var. x) is approx., but
not exactly crossing numbers of
τ_x w/ $\tau_1, \dots, \tau_n$.

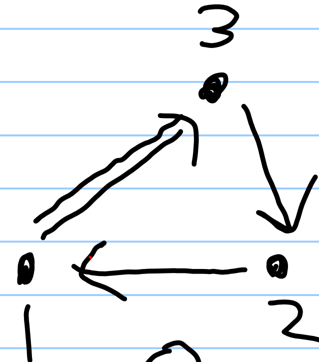
2) In general, $\{\text{denoms of cluster vars}\} \overset{\text{conjecturally}}{\longleftrightarrow} \{\text{pos real } \underline{\text{Schur}} \text{ roots}\}$

$$\{\text{pos real Schur roots}\} \subseteq \{\text{pos real roots}\}$$

(Proven when Q acyclic.)

Example 1: Let triangulation T for (S, M) be

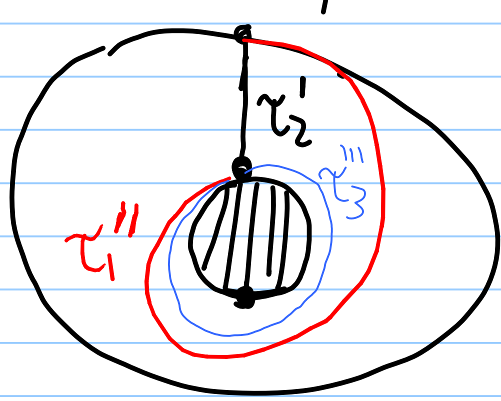




Q_T

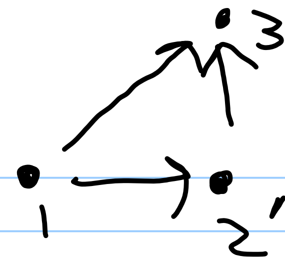
$$B_T \quad \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

Mutate by 2, 1, then 3:



Let us compute the Laurent expansion of x_3'''.

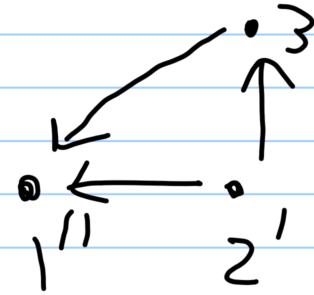
(5) $x_2 x_2' = x_1 + x_3$



$$\Rightarrow x_2' = \frac{x_1+x_3}{x_2}$$

$Q_{T'}$

$$x_1 x_1'' = x_2' x_3 + 1$$

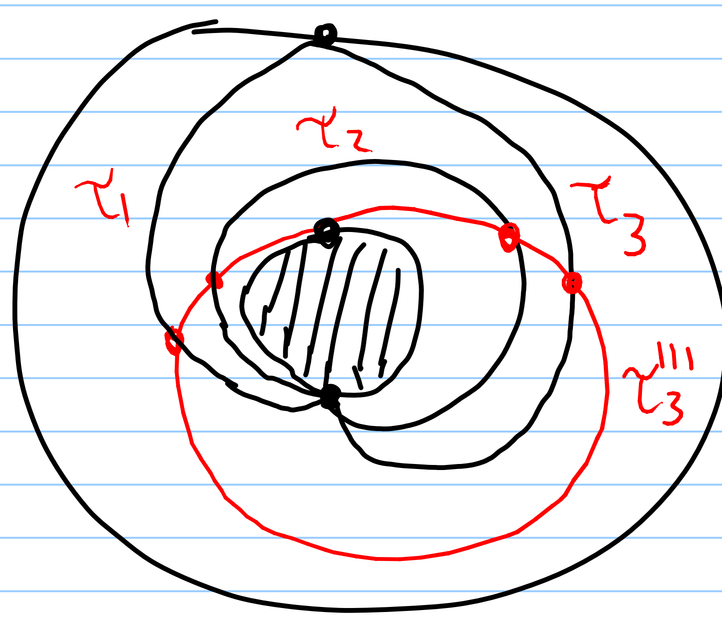


$$\Rightarrow x_1'' = \frac{x_1x_3 + x_3^2 + x_2}{x_1x_2}$$

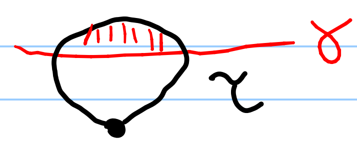
$$x_3 x_3''' = x_1'' + x_2'$$

$$\Rightarrow x_3''' = \frac{x_1x_3 + x_3^2 + x_2 + x_1^2 + x_1x_3}{x_1x_2x_3}$$

However, looking at crossing pattern



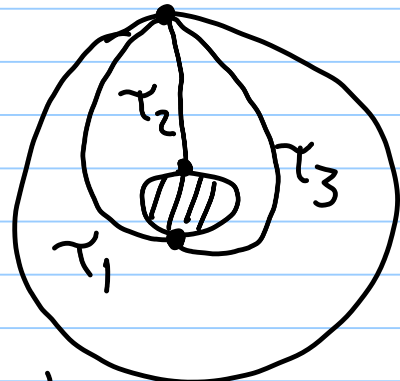
Crosses τ_2 twice, not once.

Issue: 

γ cuts out contractible sector

⑥ Example 2 :

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}$$

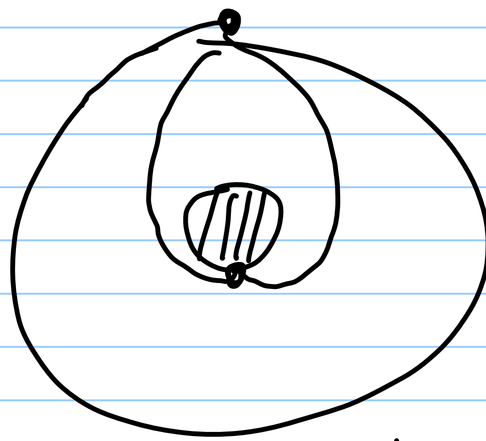


$\alpha_1 + 2\alpha_2 + \alpha_3$ is a real root, but not a Schur root.

Whether a root is Schur or not depends on orientation of Q.

Indeed, no cluster variable with denominator $x_1 x_2^2 x_3$ in this case.

Back to



$1 \rightrightarrows 2$ case

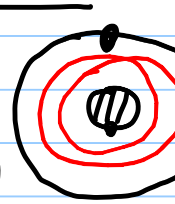
Already know $\{$pos real root$\}$ $\longleftrightarrow$ $\{$cluster variables$\}$ in this case.

Any guesses on meaning of imaginary roots in this case?

More later

(Sherman-Zelevinsky)

e.g. $(k=2)$



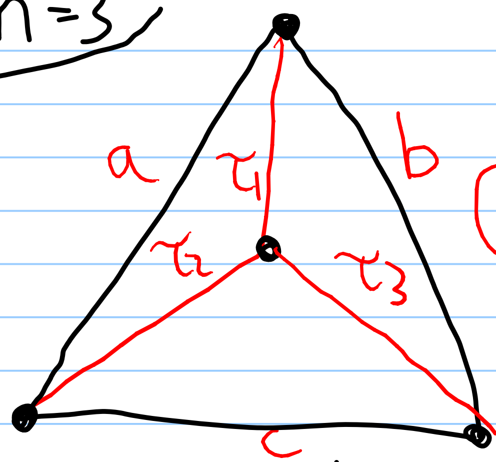
$z_k \longleftrightarrow k\alpha_1 + k\alpha_2$

⑦ Surface with punctures

(i.e. interior marked points)

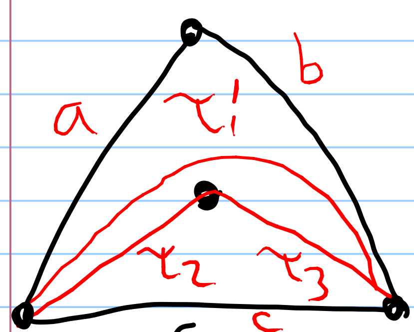
Consider a once-punctured n-gon

$n=3$



A representative triangulation (maximal collection of non-isotopic, non-crossing arcs)

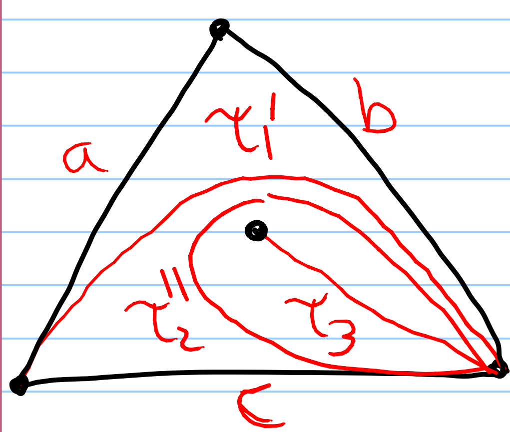
Up to rotationally symmetry, mutate/flip τ_1 $(x_1 x_1' = x_2 + x_3)$



Now flip τ_2

$x_2 x_2' = \underline{\ ?\ }$

$\downarrow \mu_2$

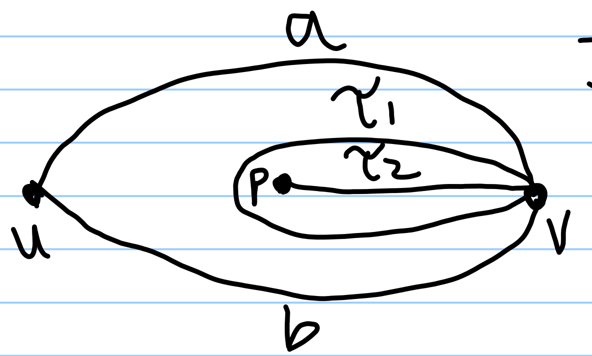


Bigger problem

How to flip τ_3 at this point ?

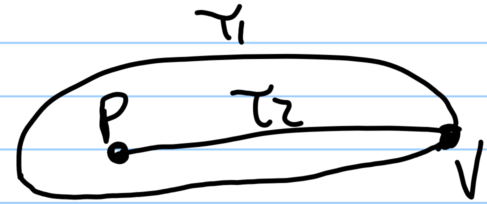
8 Fomin-Shapiro-Thurston's Sol

Tagged arcs



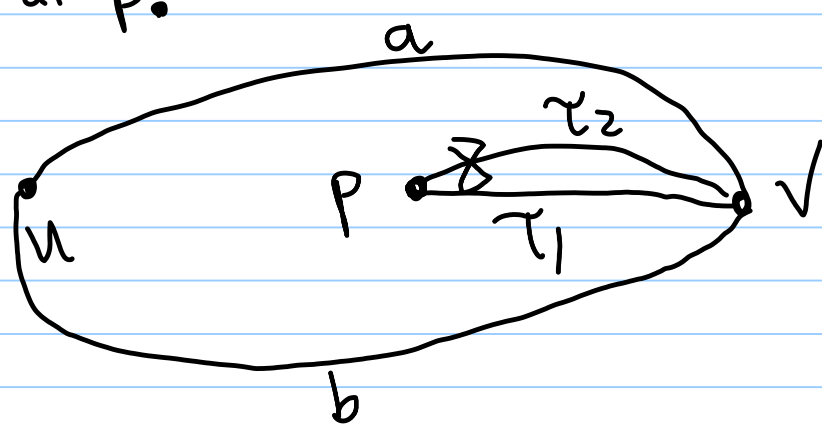
Is called a <u>self-folded quadrilateral</u>

&



is a <u>self-folded triangle.</u>

We replace loops cutting out a once-punctured monogon (e.g. τ_1) with <u>tagged arc</u> with a notch at p.



More generally, a <u>tagged arc</u> is allowed to have a <u>notch</u> at either endpoint (if endpoint is a puncture an internal marked point)

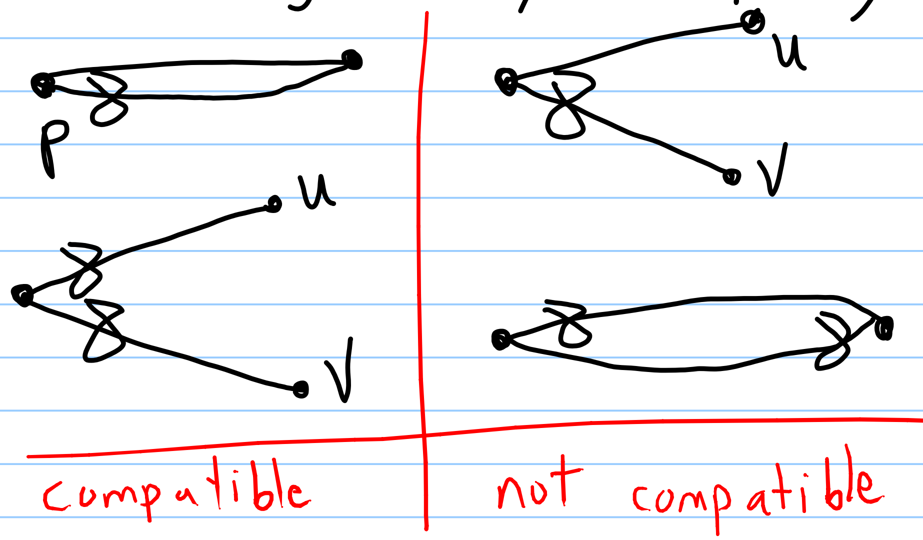
A <u>tagged triangulation</u> is a maximal collection of <u>compatible</u> tagged arcs.

9. Compatible means

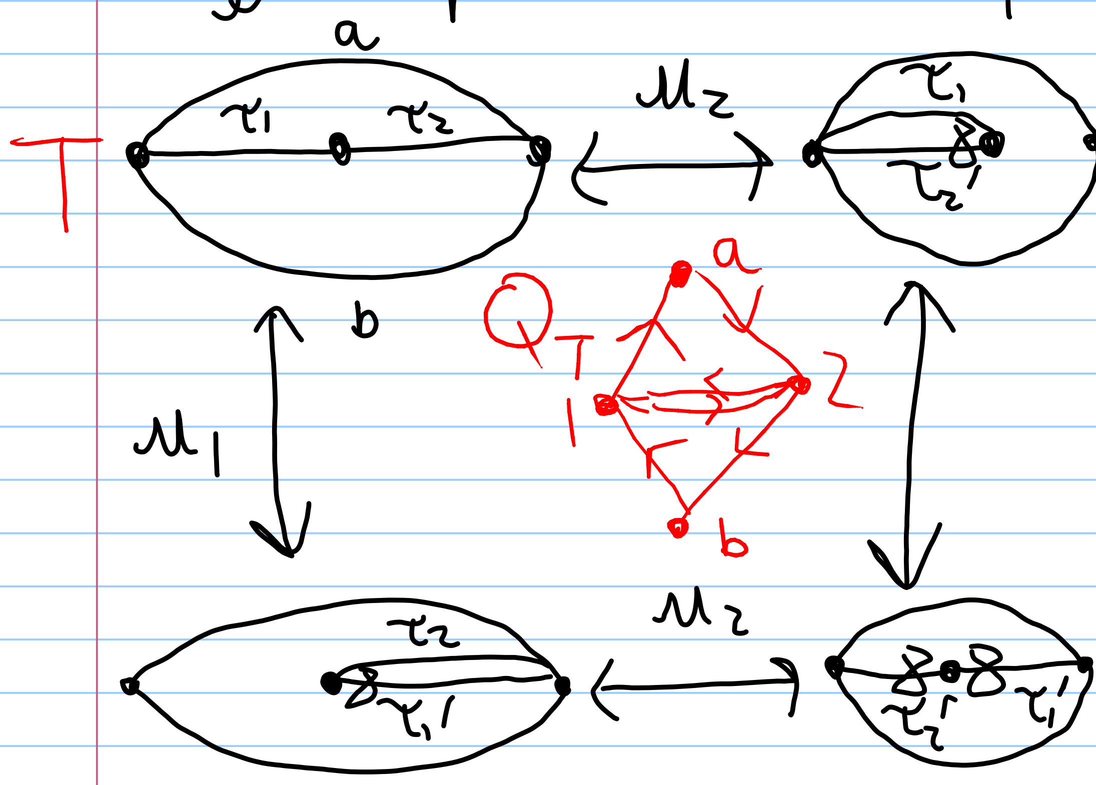
i) no two arcs cross

ii) no two arcs are isotopic unless their notching differs at exactly one endpoint

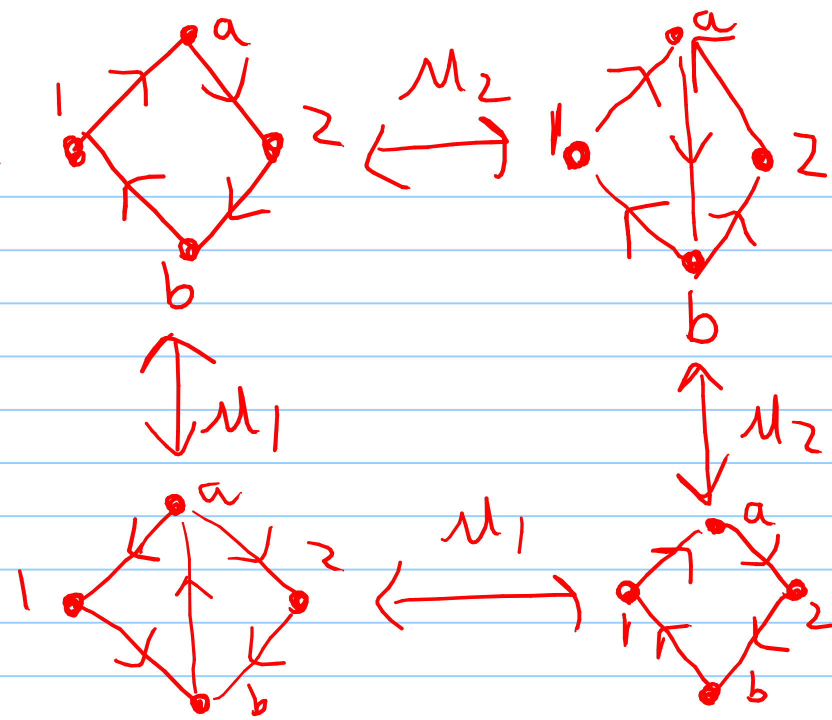
iii) at each puncture, every incident tagged arc is either unnotched or notched (w/ (ii) being the only exception)

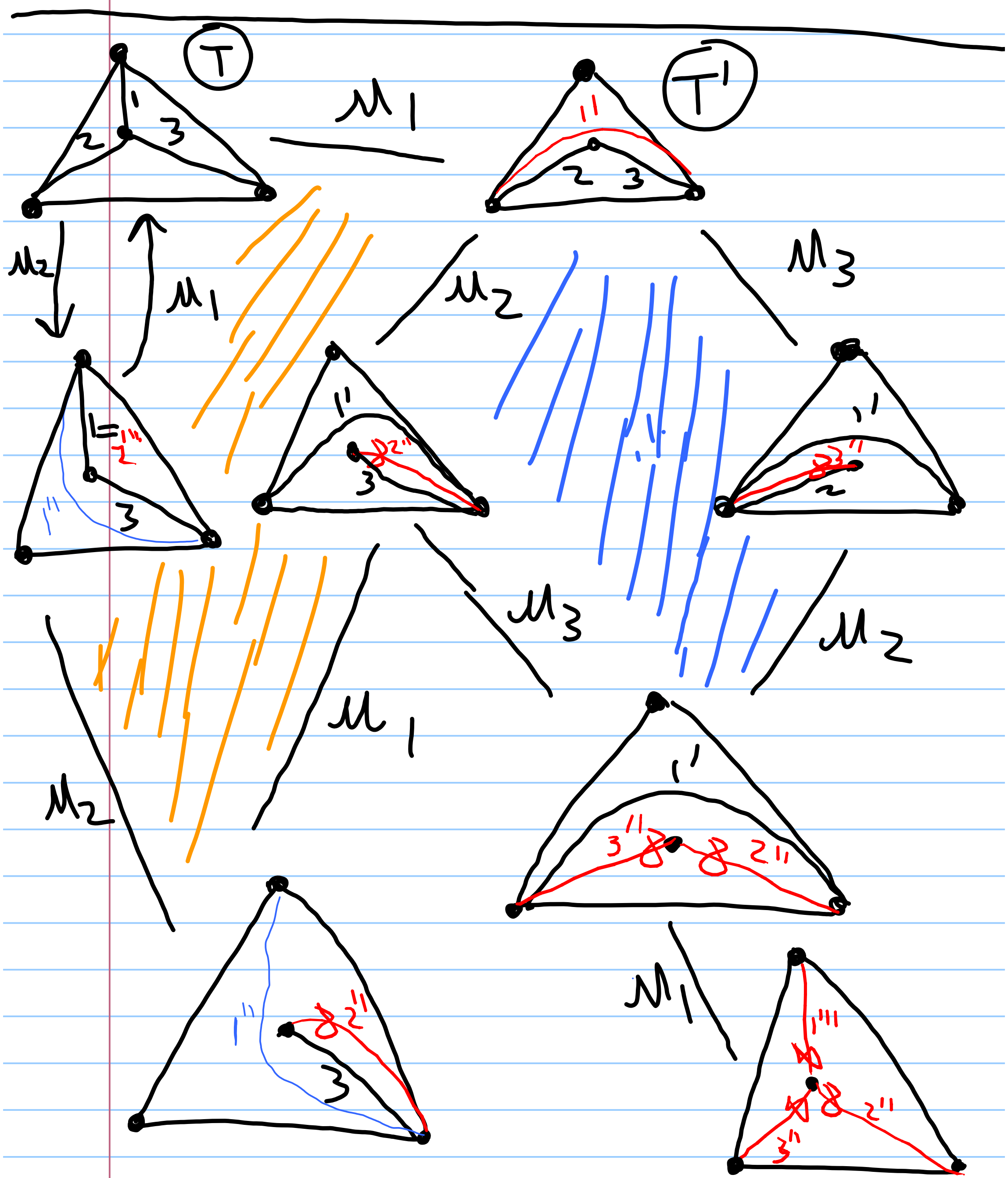


tagged flips for self-folded quad



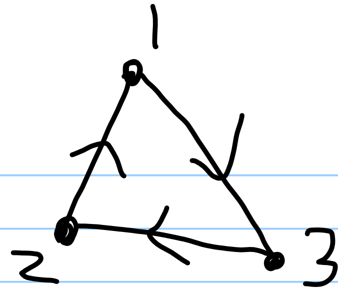
10 Q_T





11

Q_T

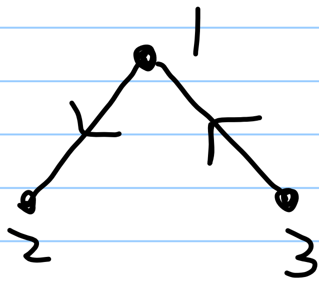


$A_3 = D_3$

Notice pentagon & quad of associahedron

μ_1

$Q_{T'}$



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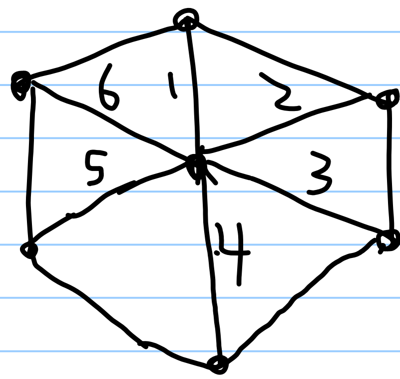
once punctured polygon



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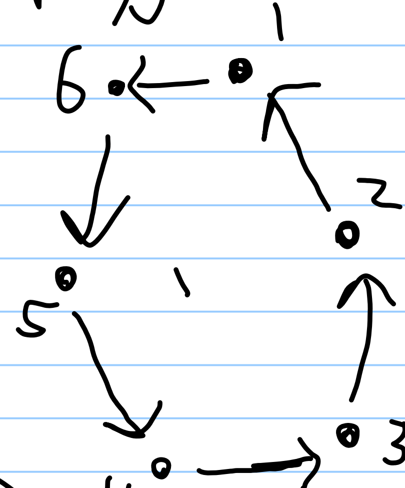
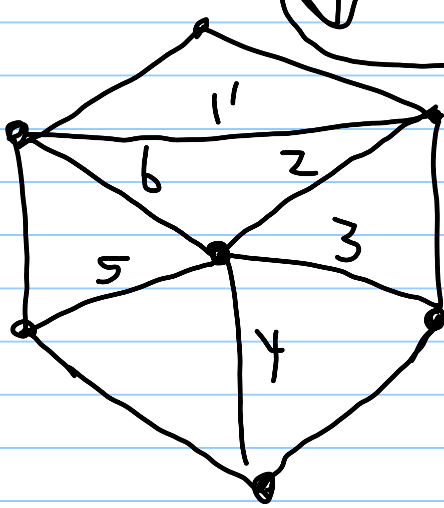
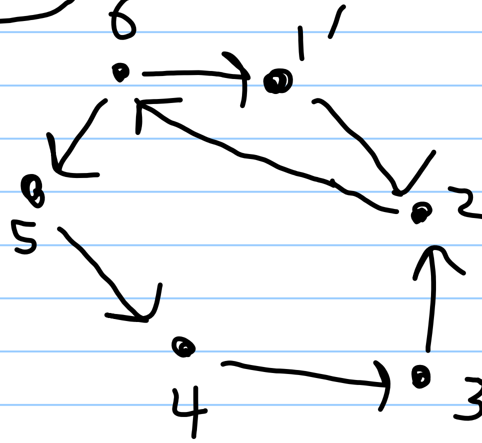


Q_T

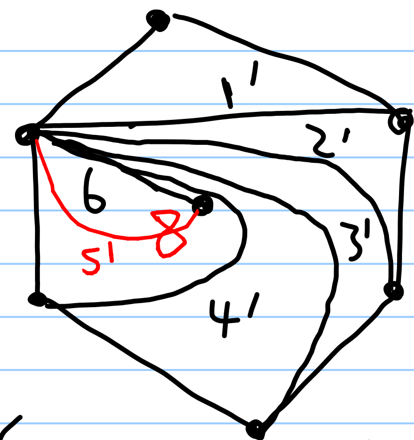
μ_1

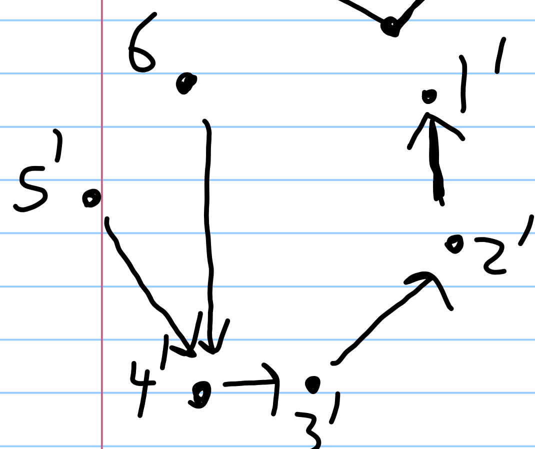


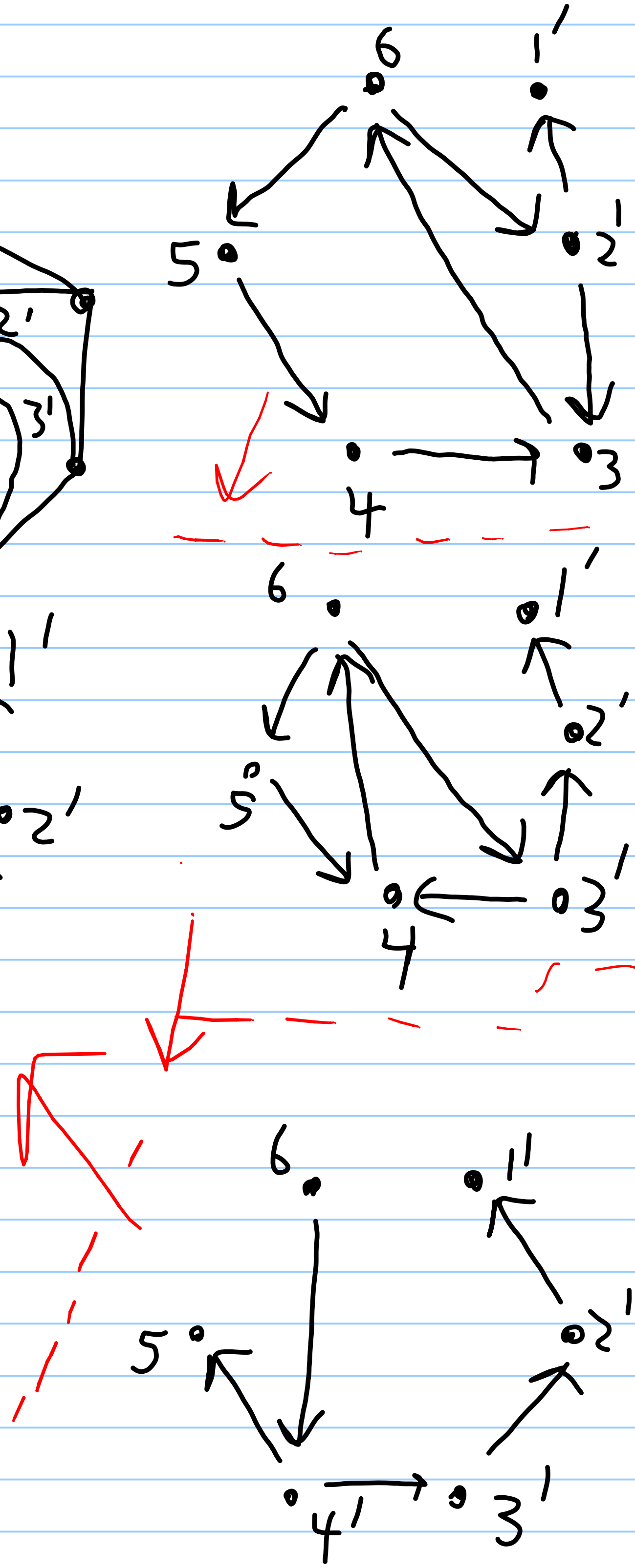


(12) Then, mutate $\mu_5 \circ \mu_4 \circ \mu_3 \circ \mu_2$:

$\mu_5 \circ \dots \mu_1(T)$

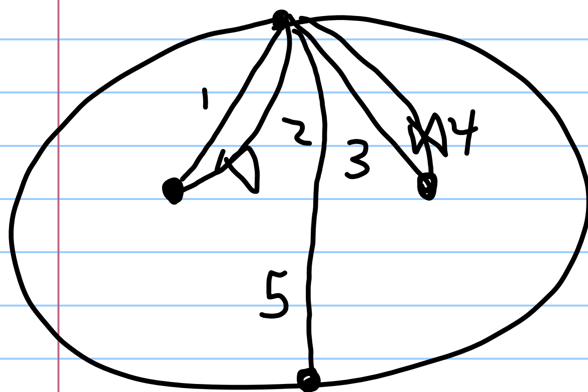


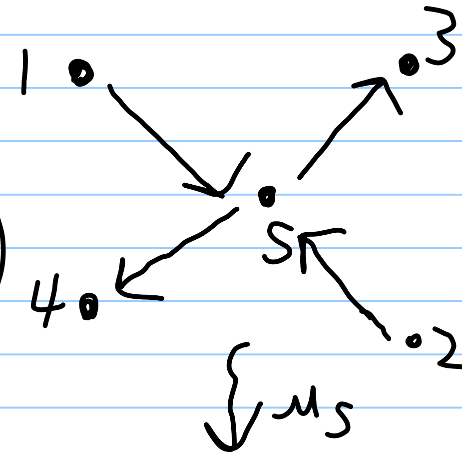




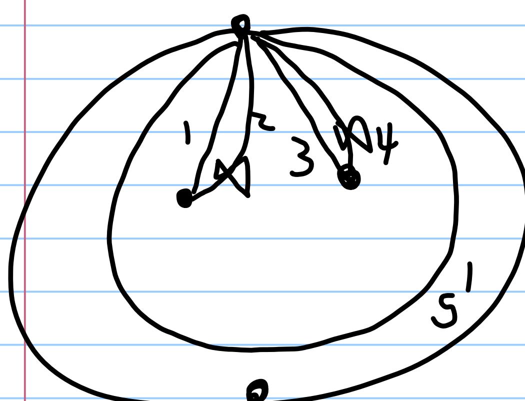
D_n type

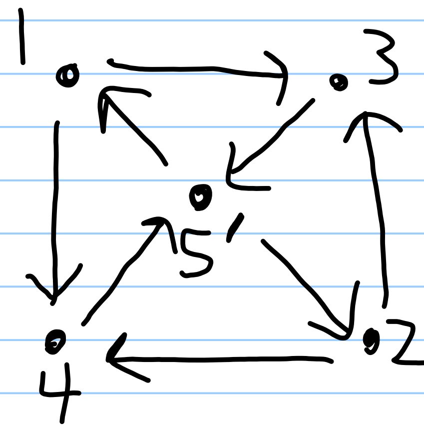
(13) <u>Similarly</u> $\tilde{D}_4$ (affine)





μ_5





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