Lecture 26: Teichmüller Theory

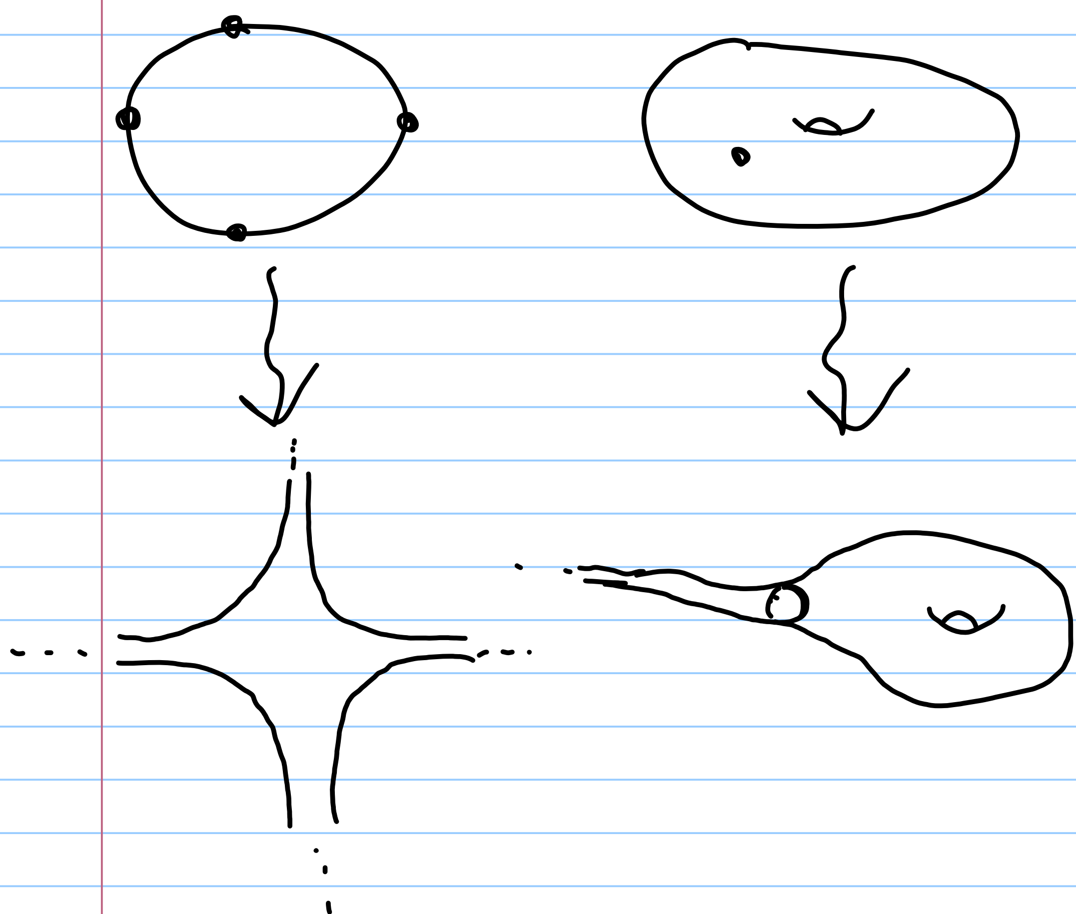
Gregg Musiker Math 8680 (4-25-11)

Note Title 4/25/2011

① Given a marked surface (S,M), $\mathcal{T}(S,M)$ Teichmüller is defined to be the space of metrics on (S,M) satisfying the following properties:

- hyperbolic (constant curvature -1)
- have geodesic boundary on ∂S
- have cusps at marked points M.

Examples



Fact: $\mathcal{T}(S,M)$ is a manifold of $\mathbb{R}$-dim $6g-6+2p+3b+c$

$\# M \cap (S - \partial S)$ $\#$ boundary comps. $\# M \cap \partial S$

(2) polygon $(n+3)$-gon

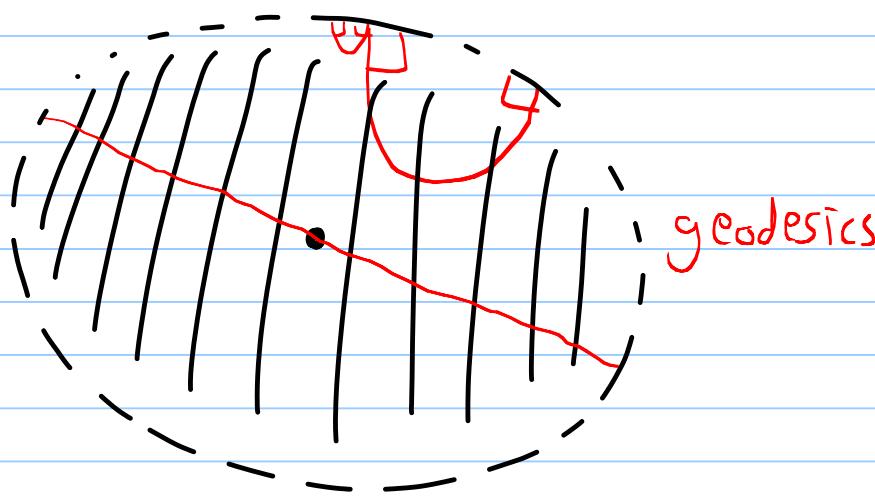
e.g. $g=0,\ p=0,\ b=1,\ c=n+3$

$\mathcal{T}(S,M)$ has $\dim = n$

Can view this polygon as Poincaré disk model for hyperbolic space.

Metric $=$ ds satisfying

$$ds^2 = \frac{dx^2 - dy^2}{(1-r^2)^2} \quad \text{with} \quad r = \sqrt{x^2+y^2}$$



Points on the boundary are infinitely far away from the center.

e.g. the distance from

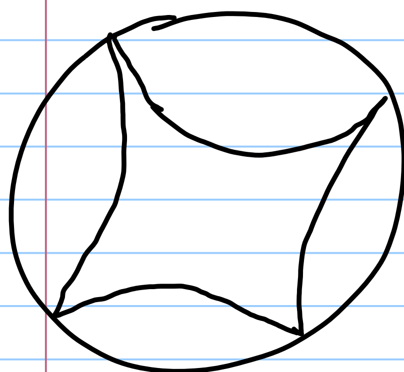
$(1,0)$ to $(0,0)$ is

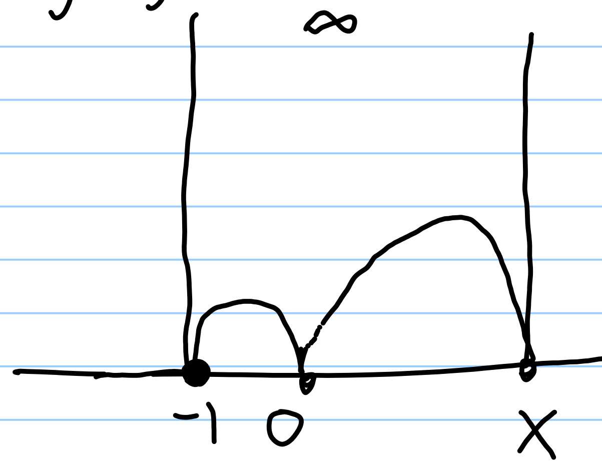
$$\int_0^1 \frac{dx}{1-x^2} = \operatorname{arctanh}(x)\Big|_0^1$$

$$= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\Big|_0^1 \to \infty.$$

③ Up to action of $PSL_2(\mathbb{R})$, we can fix 3 points on boundary and choose the other n freely.

If instead of Poincare Disk model we used upper half-plane, 3 points could be at $-1, 0, \infty$

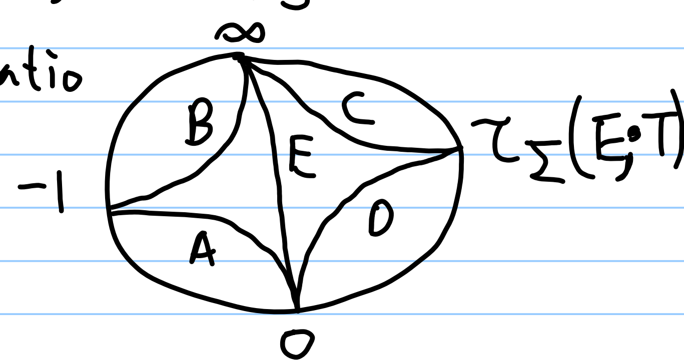




Remaining corner of a quadrilateral is free $X \in \mathbb{R}_{>0}$ but is an invariant of the quadrilateral, cross-ratio $=$ shear coordinates.

Given a hyperbolic structure $\Sigma \in \mathcal{T}(S,M)$ and a triangulation $T = \{E_i\}_{i=1}^{n}$, the shear coord.

$\tau_\Sigma(E;T)$ of edge $E \in T$ is the cross-ratio



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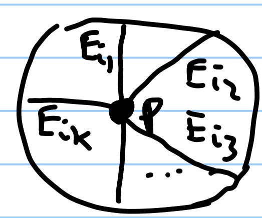
Theorem: The map $\mathcal{T}(S,M) \longrightarrow \mathbb{R}^n$

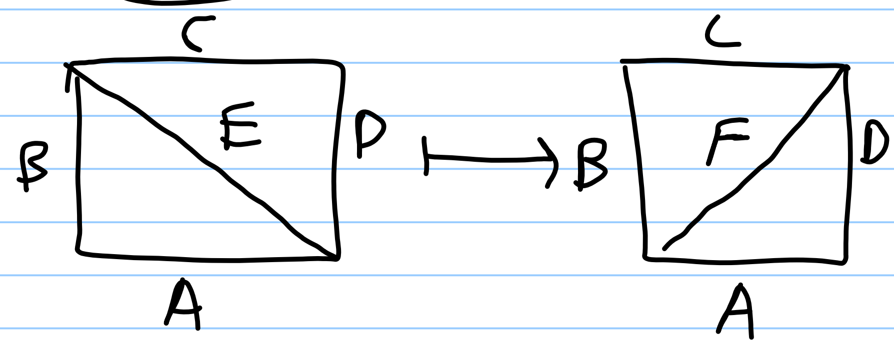
$$\Sigma \longmapsto \left\{\tau_\Sigma(E_i;T)\right\}_{i=1}^{n}$$

is a homeomorphism onto the subset of $\mathbb{R}^n$ where for each puncture p, and incident arcs $E_{i_1}, \dots, E_{i_k}$, we have

$$\prod_{j=1}^{k} \tau_\Sigma(E_{i_j};T) = 1.$$



When we flip quads to get from $T \to T'$,



shear coordinates change in a predictable way:

$$\tau_\Sigma(F;T') = \tau_\Sigma(E;T)^{-1}$$

$$\tau_\Sigma(A;T') = \tau_\Sigma(A;T)\left(1+\tau_\Sigma(E;T)^{-1}\right)^{-1}$$

$$\tau_\Sigma(B;T') = \tau_\Sigma(B;T)\left(1+\tau_\Sigma(E;T)\right)$$

$$\tau_\Sigma(C;T') = \tau_\Sigma(C;T)\left(1+\tau_\Sigma(E;T)^{-1}\right)^{-1}$$

$$\tau_\Sigma(D;T') = \tau_\Sigma(D;T)\left(1+\tau_\Sigma(E;T)\right).$$

(5) As we will see shortly, these mutations also appear in cluster alg's!

Recall, the original definition

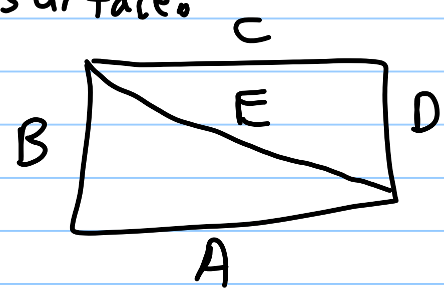
$(\mathbb{X}, \mathbb{Y}, B)$ $\mathbb{X} = \{x_1, \dots, x_n\}$ initial cluster,

$\mathbb{Y} = \{y_1, \dots, y_n\}$ initial coeffs, $B =$ $n \times n$ exchange matrix

$$x_k' x_k = \frac{y_k \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{(y_k \oplus 1)}.$$

$$y_j' = \begin{cases} y_k^{-1} & \text{if } j=k, \\ y_j \left(\frac{y_k}{y_k \oplus 1}\right)^{b_{kj}} & \text{if } j \neq k \ \& \ b_{kj} > 0, \\ y_j (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k \ \& \ b_{kj} \leq 0. \end{cases}$$

Let us focus on case of tropical semifield and cluster algebra from a surface.

Then  gives rise to

portion of exchange matrix:

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$$\begin{array}{c|ccccc} & E & A & B & C & D \\ \hline E & 0 & 1 & -1 & 1 & -1 \\ A & -1 & 0 & & & * \\ B & 1 & & 0 & & \\ C & -1 & * & & 0 & \\ D & 1 & & & & 0 \end{array}$$

Thus, if we mutate μ_E, $x_{E'} = x_F$ and

$$y_F = y_E' = y_E^{-1},$$

$$y_A' = y_A \left(\frac{y_E}{y_E \oplus 1} \right) = y_A \frac{y_E}{y_E} \left(\frac{1}{1 \oplus 1/y_E} \right)$$

$$y_B' = y_B (y_E \oplus 1)$$

$$y_C' = y_C \left(\frac{y_E}{y_E \oplus 1} \right) = y_C \frac{y_E}{y_E} \left(\frac{1}{1 \oplus 1/y_E} \right)$$

$$y_D' = y_D (y_E \oplus 1)$$

Moral: Coeff Dynamics behave like Tropical shear coordinate dynamics.

This motivated use of laminations.

Def: An elementary lamina is like an arc except instead of going between marked points, its endpoints intersect the boundary (away from a marked point) or spiral into a puncture, or is a closed curve.

More precisely, an integral unbounded measured lamination on a marked surface (S, M) is a finite collection of pairwise non-intersecting curves, each of which has no self-intersections, modulo isotopy (relative to M) such that:

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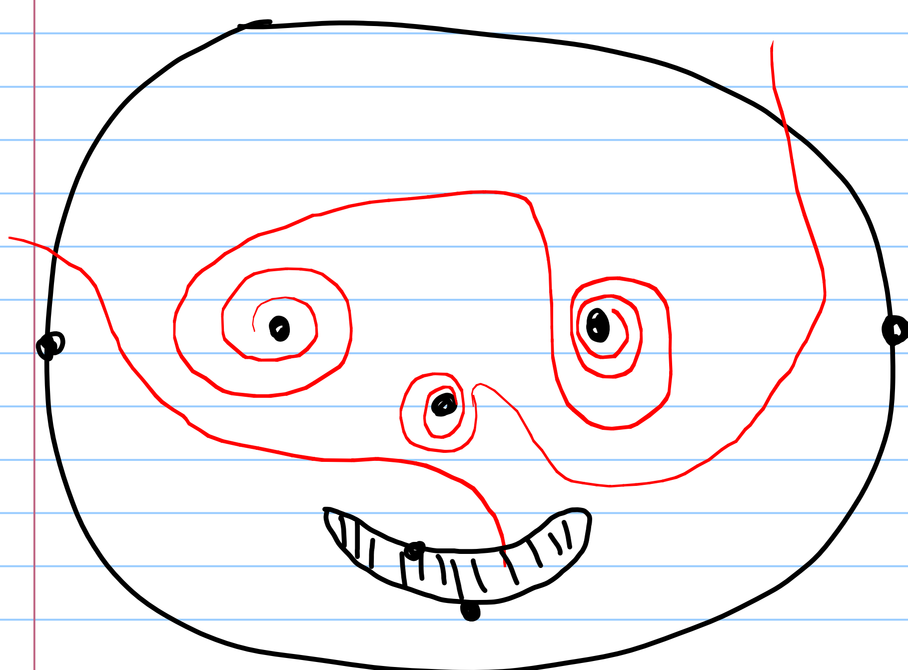
i) a closed curve
ii) a curve connecting two unmarked points on ∂S
iii) a curve with one endpoint on $\partial S - M$ and one endpoint spiraling into a puncture (clockwise OR counter-clockwise), or
iv) a curve with both endpoints spiralling into a puncture.

The following curves are disallowed:

i) a closed curve bounding an unpunctured or once punctured disc,
ii) a curve with two endpoints on ∂S which is isotopic to a boundary arc containg 0 or 1 marked point



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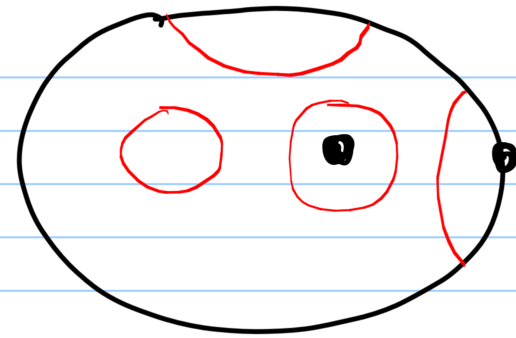
Disallowed



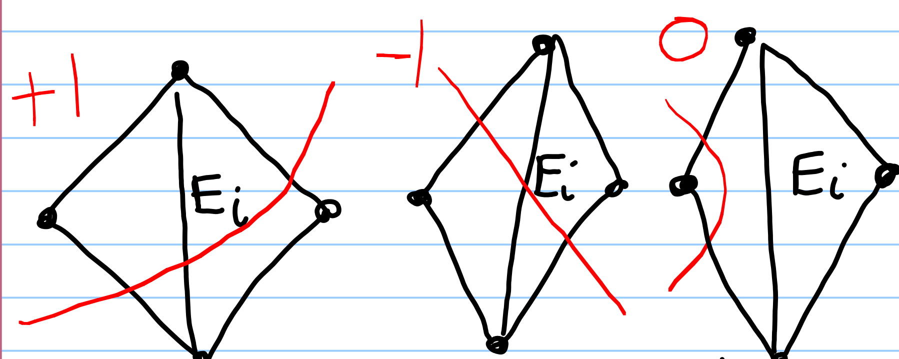
∂S

We assign a shear coordinate $\epsilon^{\mathbb{Z}}$ to each arc $E_i \in T$ of a triangulation w.r.t. a choice of lamination:

$$b_{E_i}(T, L) \quad \text{for each } E_i \in T.$$

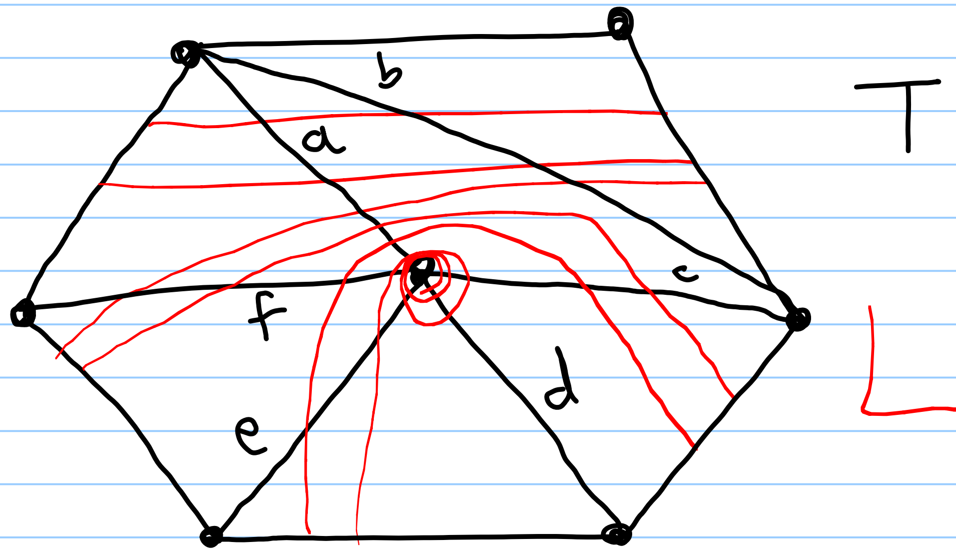
As above, we look at quadrilateral inscribing E_i (in triangulation L)

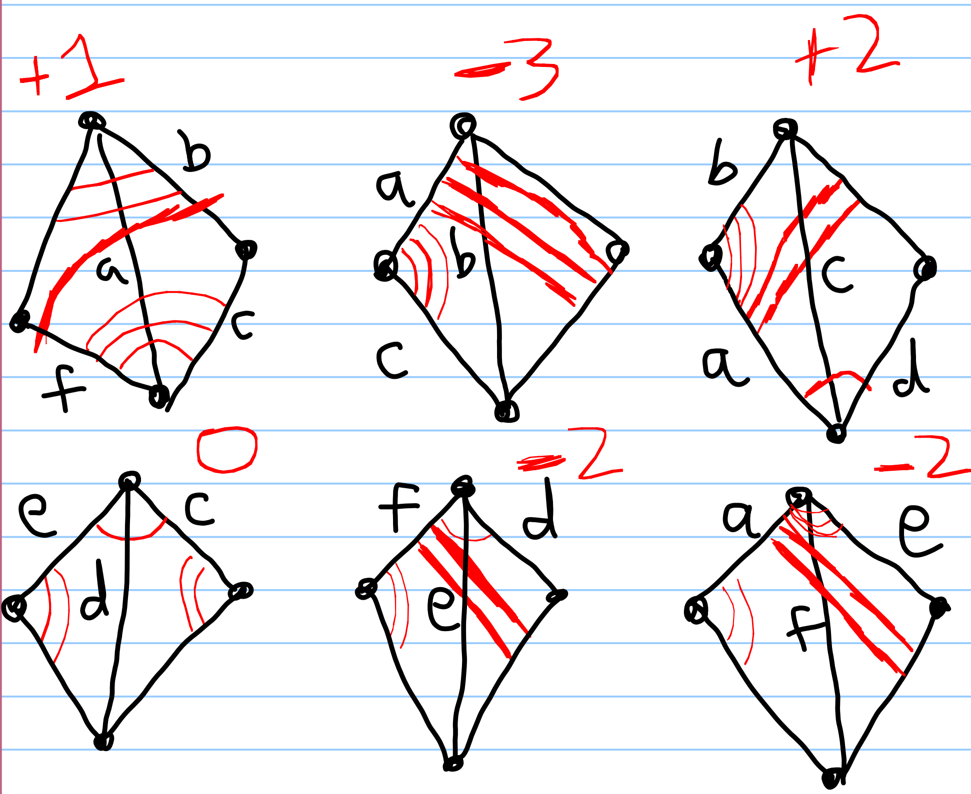
For each curve of lamination L cutting through the quadrilateral, we calculate a contribution to the shear coordinate. Adding them all up gives contribution



\$ Zilch and all other crossings of adjacent sides give zero too.

⑨ Example (Fig 3 of Fomin-Thurston)

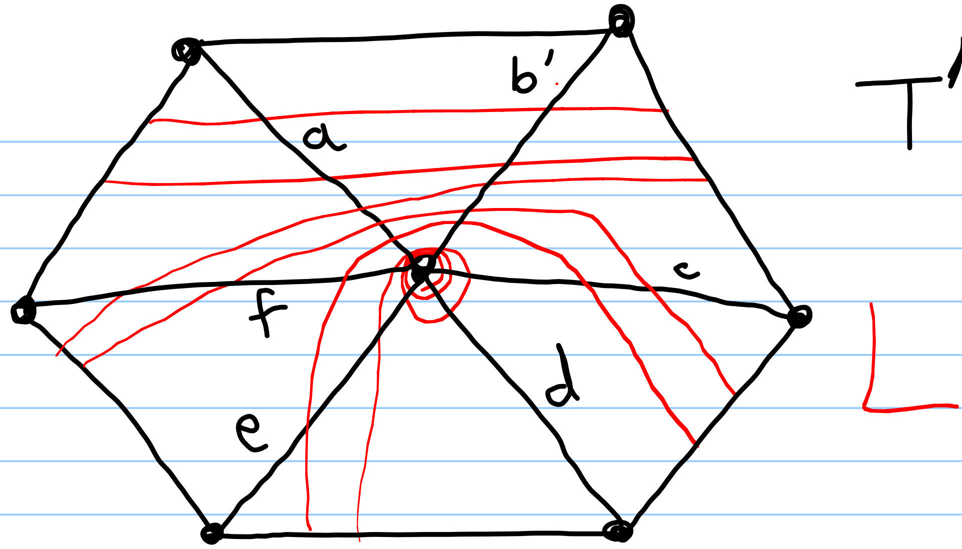


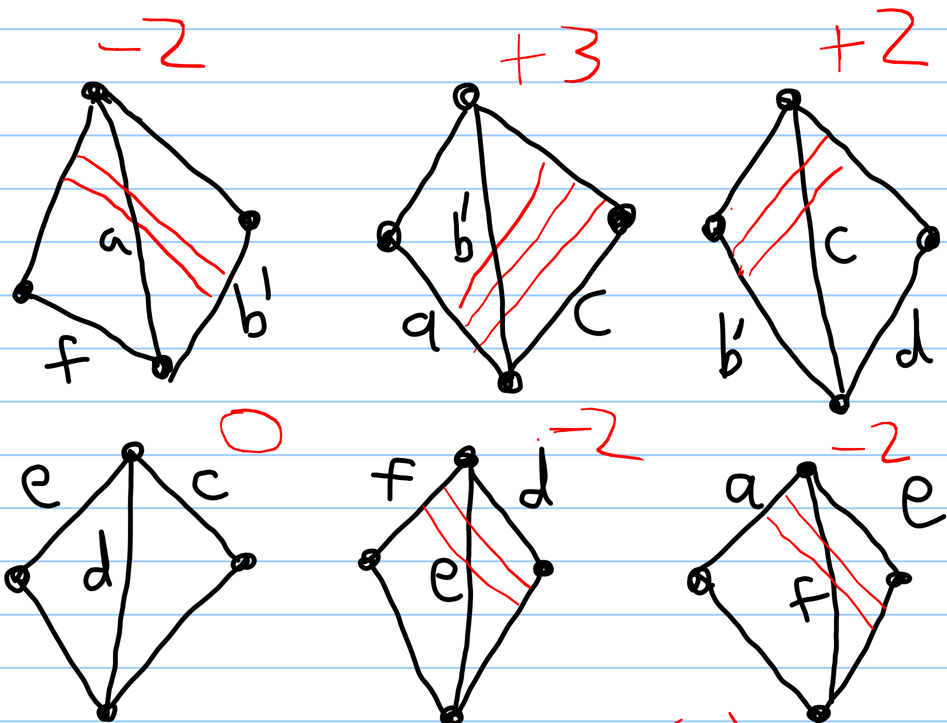


$b_a(T, L) = 1, \; b_b(T, L) = -3, \ldots$

Let us now flip $b \mapsto b'$ to get triangulation T' with L fixed:

(10)





$$+3 = -(-3)$$

Notice: $b_{b'}(T',L) = -b_b(T,L)$

$$-2 = 1 - \max(-(-3),0)$$

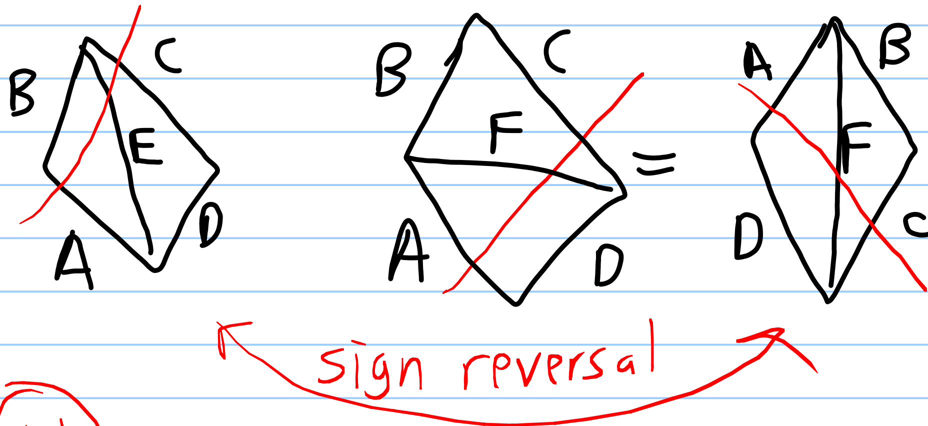
$$b_a(T',L) = b_a(T,L) - \max(-b_b(T,L),0)$$

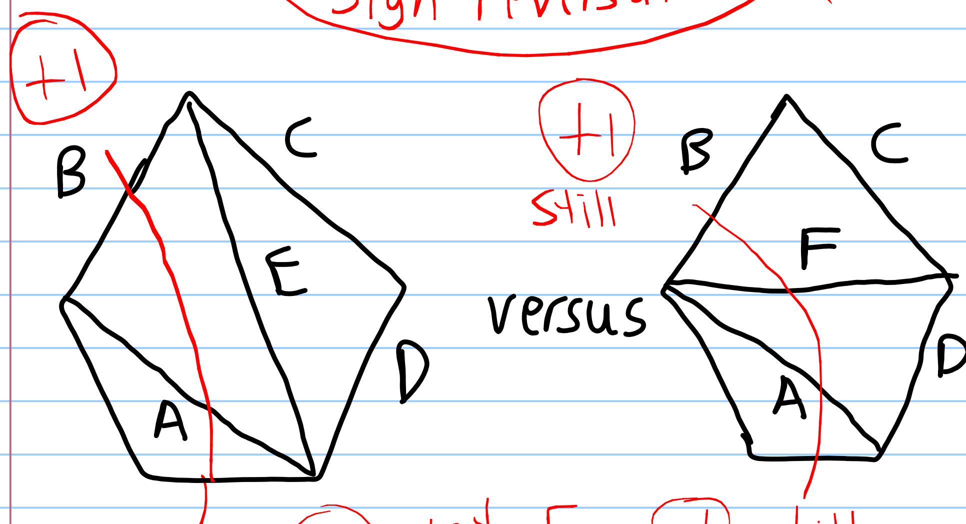
$$+2 = 2 + \max(-3,0)$$

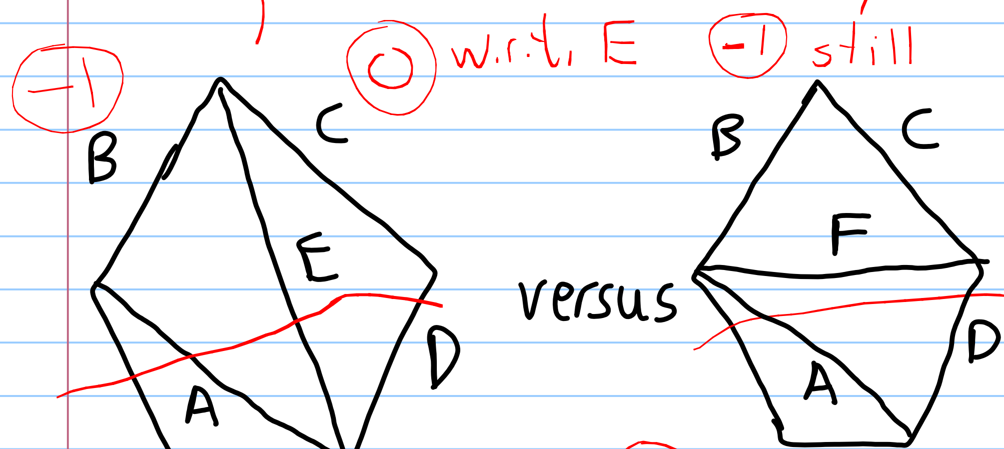
$$b_c(T',L) = b_c(T,L) + \max(b_b(T,L),0)$$

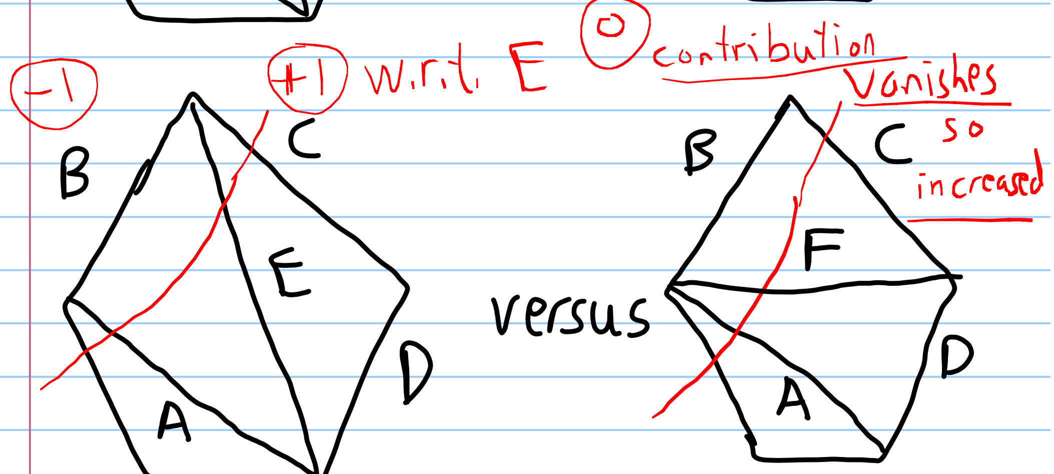
$$b_\tau(T',L) = b_\tau(T,L) \quad \text{o.w.}$$

(11) Not just this example, but logic works:

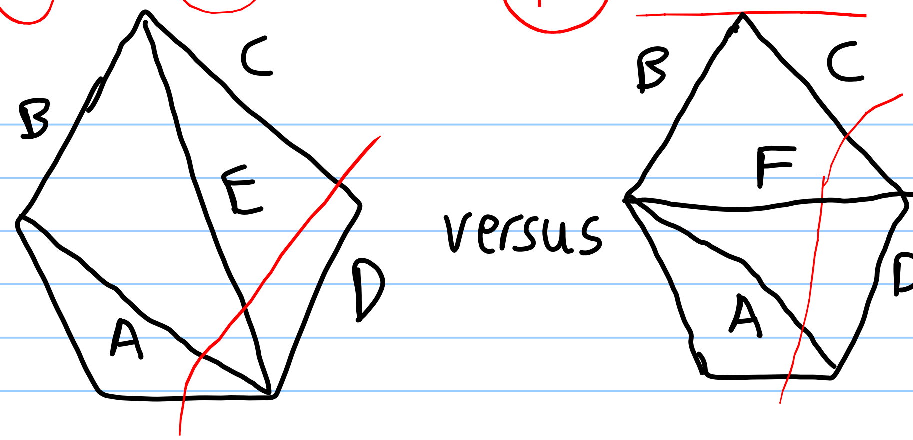








0 +1 w.r.t E +1 increased



$$b_A(T', L) = b_A(T, L) + \underbrace{b_E(T, L)}_{\text{if } > 0}$$

$$= b_A(T, L) + \max(b_E(T, L), 0).$$

✓

Moral: if we let $\oplus = \max(-,-)$, coefficient dynamics agree with computing shear coordinates with laminations.

Thus, we can use laminations to build an arbitrary cluster algebra of geometric type (with $(m+n) \times n$ exchange matrix) as long as $n \times n$ top corresponds to a cluster algebra from a surface.

Thm (W. Thurston) For a fixed triangulation T without self-folded triangles, the map

$$L \to \left(b_E(T, L)\right)_{E \in T}$$

is a bijection between integral unbounded measured laminations and $\mathbb{Z}^n$.

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Consequently, if T has no self-folded triangles, any desired coeff. pattern can be achieved by m Laminations. Each L_i corresponds to a row of coefficients.

<u>Thm</u> (W. Thurston, Fock, Goncharov)

If T, T' are triangulations without self-folded triangles are related by flipping edge E_k, then $(m+n) \times n$ exch. matrices $\tilde{B}(T, L)$ & $\tilde{B}(T', L)$ are related by μ_k.

Fomin-Thurston defined <u>shear coordinates for triangulations with self-folded triangles</u> or tagged arcs too.

Enlarges bijection and Theorem.

<u>Next time</u>: Lambda Lengths, a Teichmüller interpretation of cluster variables.