Last time, we described laminations and how we can obtain general $(m+n) \times n$ exchange matrices s.t. $n \times n$ top matrix comes from a surface.

E.g.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & -1 & 0 \\
2 & 1 & 0 & -1 \\
3 & 0 & 1 & 0 \\
4 & -1 & 0 & 1 \\
L_1 & 1 & 1 & 0 \\
L_2 & 0 & 0 & 1 \\
L_3 & 2 & 0 & 0 \\
\end{bmatrix}
\]

Recall the (non-trivial) Thm

any row in $\mathbb{Z}^n$ can be achieved by the choice of some lamination.

A special case: Principal coeffs

For any tagged arc $\gamma$ in a triangulation, we define the elementary lamination $L_\gamma$ associated to $\gamma$ to be the single curve obtained from $\gamma$ by changing $\gamma$'s endpoints in one of the following 3 ways:
2. If \( \gamma \)'s endpoint \( q \) lies on a boundary \( B \), we move \( q \) slightly "counter-clockwise" around \( B \) so that it misses marked points.

If \( \gamma \)'s endpoint \( p \) lies at a puncture \( \ast \) and \( \gamma \) is tagged plain, we rotate \( L \gamma \) counter-clockwise around \( p \):

Finally, if \( \gamma \) is notched at \( p \), then we rotate \( L \gamma \) clockwise around \( p \) instead:

Putting these rules together:
Example:

```
3

```

gives laminations

```
4

```

No lamination crosses $\gamma_4$ so crossing quadrilateral in negative formation impossible.

only $L_1$ crosses $\gamma_1$, $\gamma_1$, and $\gamma_3$ to $\gamma_1$

so column corresponding to $\gamma_1$ is

\[
\begin{bmatrix}
L_1 & L_2 & L_3 & L_4 & L_5 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}^T
\]
Similarly, all five rows/cols give $I_5$.

We now switch gears and talk about Teichmüller interpretations of cluster variables/arcs.

**Def:** A horocycle, at an ideal point $p$, is a set of points which are all equidistant to $p$.

Topologically:  

But in lift to hyperbolic upper half plane or the Poincare disk,

**Def:** The decorated Teichmüller space $\tilde{J}(S_jM)$ is parametrized by data consisting of a point in $J(S_jM)$ and a choice of horocycle around each cusp from $M$.

**Def (Penner):** For an arc $E$ on $(S_jM)$ and a choice $\Sigma \in \tilde{J}(S_jM)$, the length $\ell \Sigma(E) \overset{\circ}{=} \text{length of the geodesic rep. of } E$
between intersections with horocycles.

If horocycles chosen large enough so that they intersect $l_\Sigma(\gamma)$ is negative instead.

In topological viewpoint

In this e.g.

$l_\Sigma(\alpha_2)$ is negative.

Def: The $\lambda$-length of $E$ is defined as $\lambda_\Sigma(E) = e^{l_\Sigma(E)/2} \in \mathbb{R}_{>0}$.

Penner coordinates for decorated Teichmüller space:

Thm (Penner): For any triangulation $T = \{E_i\}_{i=1}^n$ without self-folded triangles, the map $\Pi \lambda(\gamma) : \mathcal{F}(S,M) \to \mathbb{R}_{>0}$ is a homeomorphism.
Sketch of Proof: If our surface was a single triangle with fixed vertices, then choosing \( l_\Sigma(A), l_\Sigma(B), l_\Sigma(C) \) 

\[
\begin{align*}
\lambda_\Sigma(\cdot) \in (-\infty, \infty) \quad \text{so} \quad \lambda_\Sigma(\cdot) \in \mathbb{R}_{>0}
\end{align*}
\]

uniquely determines the decoration with no cycles,

\[\text{e.g. if } l_\Sigma(A), l_\Sigma(B) \text{ held fixed but } l_\Sigma(C) \text{ were increased}\]

\[\Rightarrow \text{ Lengths on all triangles determines decorated triangles and a unique way to glue adjacent triangles.}\]

Also from \( \lambda \)-lengths (or \( l_\Sigma(E) \)) for all arcs of a triangulation + boundary, we can obtain lengths of any arc in surface.
**Ptolemy Relation**

For any ideal quadrilateral and $\Sigma \in \mathcal{J}(S,M)$, we have

$$\lambda_\Sigma(E) \lambda_\Sigma(F) = \lambda_\Sigma(A) \lambda_\Sigma(C) + \lambda_\Sigma(B) \lambda_\Sigma(D)$$

Not just algebraic statement, but statement about exponentials of these hyperbolic lengths.

Notice that this is a "tropical-like statement about lengths"

$$e^{\frac{l_\Sigma(E)}{2}} e^{\frac{l_\Sigma(F)}{2}} = e^{\frac{l_\Sigma(A)}{2}} e^{\frac{l_\Sigma(C)}{2}}$$

$$+ e^{\frac{l_\Sigma(B)}{2}} e^{\frac{l_\Sigma(D)}{2}}$$

$$\Rightarrow l_\Sigma(E) + l_\Sigma(F) = \log \left( e^{\frac{l_\Sigma(A)}{2}} + e^{\frac{l_\Sigma(C)}{2}} + e^{\frac{l_\Sigma(B)}{2}} + e^{\frac{l_\Sigma(D)}{2}} \right)$$

**Point:** Let $X_{E_i} \overset{\text{def}}{=} \lambda_\Sigma(E_i)$ for each $E_i \in \mathcal{T}$ a triangulation with no self-intersecting triangles plus boundary arcs. Then, choice of $(X_{E_i})'$s (as in $\mathbb{R}_{>0}$) uniquely determines data $\Sigma \in \mathcal{J}(S,M)$ for all other $\lambda_\Sigma(\Sigma)$ for $\Sigma$ another arc of $(S,M)$. Determined.
By iterations of Ptolemy Relations each \( \lambda \Xi(\mathcal{S}) \) is a Laurent polynomial in the \( X E_i \)'s, and can be thought of as a function acting on points of \( \tilde{\mathcal{F}}(S_j \mathcal{M}) \).

These are the cluster variables.

Relation to shear coordinates

Given a hyperbolic structure (undecorated) \( \Sigma \in \mathcal{F}(S_j \mathcal{M}) \) and triangulation \( T=\{ E_i \} \)

\( \tau_{\Sigma}(E_j T) = \text{cross ratio} \)

We can lift to an element \( \tilde{\Sigma} \in \tilde{\mathcal{F}}(S_j \mathcal{M}) \) by choosing horocycles and then

\[ \tau_{\tilde{\Sigma}}(E_j T) = \lambda_{\tilde{\Sigma}}(A) \lambda_{\tilde{\Sigma}}(C) \]

\[ \lambda_{\tilde{\Sigma}}(B) \lambda_{\tilde{\Sigma}}(D) \]

Does not depend on lift \( \tilde{\Sigma} \), i.e., choice of horocycles.
Notice, if we make a horcycle bigger/smaller, it affects consecutive sides of quadrilateral equally =) numerator and denominator equally.

\[ \Xi (E_{\Sigma \gamma}) \]

Question came up last time:

Why do \( l_{\Xi^\gamma} (A), \ldots, l_{\Xi^\gamma} (D) \) not determine \( l_{\Xi^\gamma} (E) \)?

Answer: shrinking horocycles on left and right while expanding horocycles on top and bottom can keep \( l_{\Xi^\gamma} (A), \ldots, l_{\Xi^\gamma} (D) \) fixed while shrinking \( l_{\Xi^\gamma} (E) \).
Some more facts about $\lambda$-lengths

Note that since horocycles lie away from cusps, their hyperbolic lengths, or lengths between intersections with arcs of a triangulation, are well-defined.

We denote these as $L(h)$ or $L(h_c)$ etc.

Lemma: In a decorated triangle, we have the identity

$$L(h_a) = \frac{\lambda \Sigma(A)}{\lambda \Sigma(B) \lambda \Sigma(C)}$$

Sketch of Proof: As $\Sigma(h_a')$ gets bigger, $\lambda \Sigma(B)$ and $\lambda \Sigma(C)$ get smaller.

If $L(h_b)$ changed, $\lambda \Sigma(A)$ and $\lambda \Sigma(C)$ grow proportionally.

Integrating metrics makes arguments rigorous.
Adding up contributions from triangles, we get

**Cor:** Around a puncture \( p \) incident to \( \tau_j, \ldots, \tau_k \) with opposite edges labeled as above,

\[
L(h_p) = \sum_{i=1}^{k} \lambda \Sigma(\tau_i) \lambda \Sigma(\tau_{i+1})
\]

[letting \( \tau_{k+1} \) denote \( \tau_1 \)]

\[
= \frac{\sum_{i=1}^{k} \sigma(1) \lambda \Sigma(\tau_1) \lambda \Sigma(\tau_3) \cdots \lambda \Sigma(\tau_k)}{\lambda \Sigma(\tau_1) \cdots \lambda \Sigma(\tau_k)}
\]

where \( \sigma \) is cycle \((12\ldots k)\).

**Def:** Define two horocycles \( h \) and \( h' \) around \( p \) to be conjugate if \( L(h') = 1/L(h) \).

**Cor:** In a once-punctured monogon \( l \) with conjugate horocycles \( h_p \) and \( h_p' \), then

\[
L(h_p) = \lambda \Sigma(l) \frac{\lambda \Sigma(r)^2}{\lambda \Sigma(r) \cdot \lambda \Sigma_1(r)}
\]

regardless of choice of \( \Sigma \in \mathcal{F}(\mathfrak{s}m) \) has \( h_p' \) instead of \( h_p \).
12. **Proof:** First statement follows from earlier Lemma.

For second, notice \( \lambda \Sigma' (l) = \lambda \Sigma (l) \) as \( l \) does not intersect \( h_p \) nor \( h_p' \).

Since \( h_p' \) defined such that

\[
L(h_p') = \frac{1}{L(h_p)}, \text{ we have}
\]

\[
L(h_p') = \frac{\lambda \Sigma'(l)}{\lambda \Sigma'(r)^2} \quad \&
\]

\[
L(h_p') = \frac{\lambda \Sigma(r)^2}{\lambda \Sigma(l)} = \Rightarrow
\]

\[
\lambda \Sigma'(l) \lambda \Sigma'(l) = \lambda \Sigma(r)^2 \lambda \Sigma'(r)^2.
\]

\[
\Rightarrow \quad \frac{\lambda \Sigma(l)^2}{\lambda \Sigma'(l)^2}. \quad \text{Taking square-roots of both sides}
\]

\[
\text{Today's work finishes the proof.}
\]

---

**Moral:** Combinatorially, if we have a bigon

\[
\begin{array}{c}
\text{By Ptolemy Relation} \\
\sum (c_1) \sum (l_1) = (\sum (a) + \sum (b)) \sum (c_2) \\
\end{array}
\]

Dividing both sides by \( \sum (c_2) \).
we now get

\[
\lambda_T(\tau) = \lambda_{\tau_1}(a) + \lambda_{\tau_2}(b)
\]

\[
\implies \lambda_{\tau_2} \text{ behaves like tagged arc}
\]

This allowed Fomin-Thurston to think of cluster variables corr. to tagged arcs as

"Take \( \Sigma \in \hat{\mathcal{F}}(s, M) \), if \( \tau \) notched at \( p \) or \( q \), change decoration \( \Sigma \) by changing horocycle at \( p \) and/or \( q \) to its conjugate \( \Sigma' \).

Then \( x_\Sigma = x(\tau) \cdot \Sigma' \).

Prop: \( \lambda_T(\tau) \cdot \lambda_T(\lambda_\Sigma) \).\]