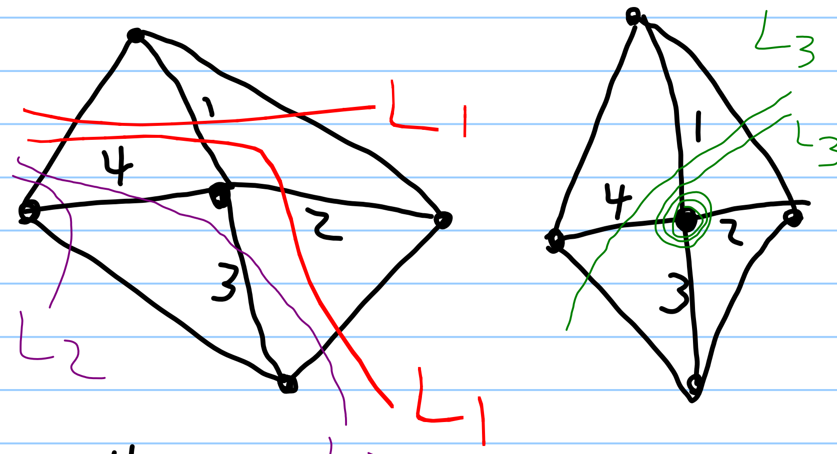
Lecture 27: Lambda Lengths (4-27-11)

Gregg Musiker Math 8680

(1) Last time, we described laminations and how we can obtain general $(m+n) \times n$ exchange matrices s.t. $n \times n$ top matrix comes from a surface.

e.g.



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ L_1 \\ L_2 \\ L_3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ \hline -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 2 & 0 & 0 & -1 \end{array}\right] \end{array}$$

Recall, the (non-trivial) Thm

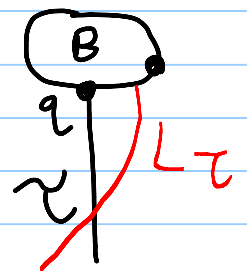
any row in $\mathbb{Z}^n$ can be achieved by the choice of some lamination.

A special case: Principal coeffs

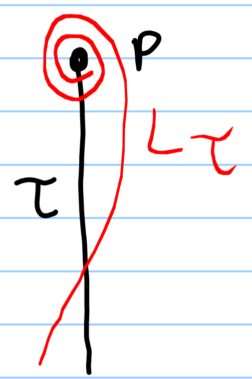
$$\left[\begin{array}{c} B \\ \hline I_n \end{array}\right]$$

For any tagged arc τ in a triangulation, we define the elementary lamination L_τ associated to τ to be the single curve obtained from τ by changing τ's endpoints in one of the following 3 ways:

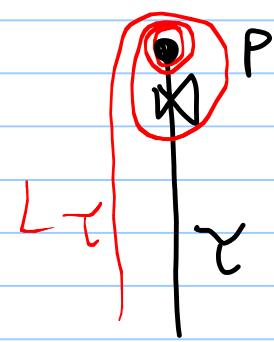
② If τ's endpoint q lies on a boundary B, we move q slightly "counter-clockwise" around B so that it misses marked points:



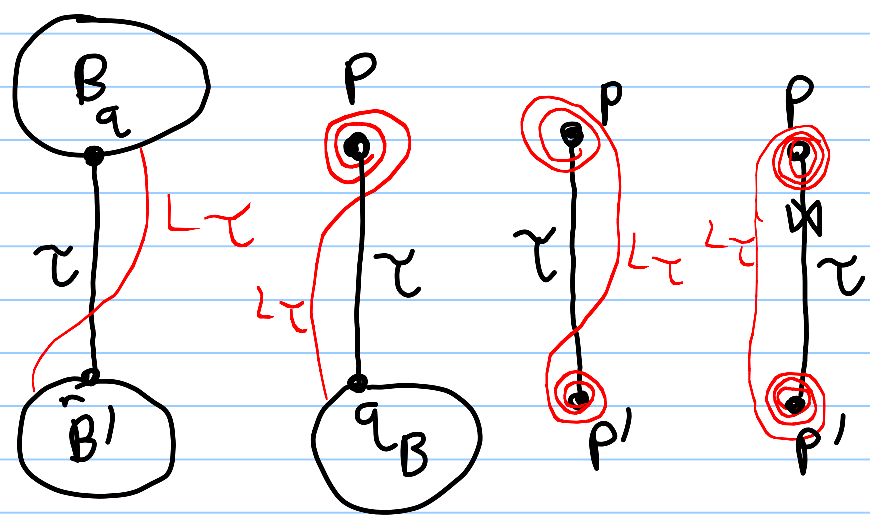
If τ's endpoint p lies at a puncture & τ is tagged plain, we rotate $L\tau$ counter-clockwise around p:



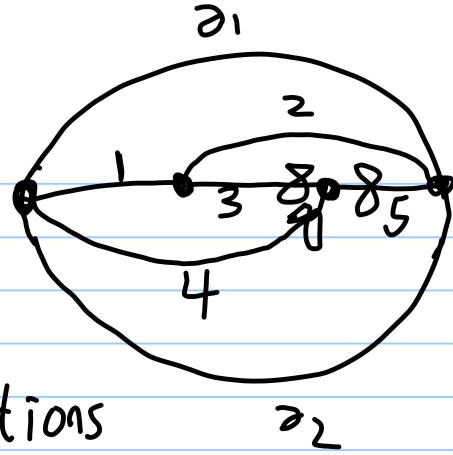
Finally, if τ is notched at p, then we rotate $L\tau$ clockwise around p instead:



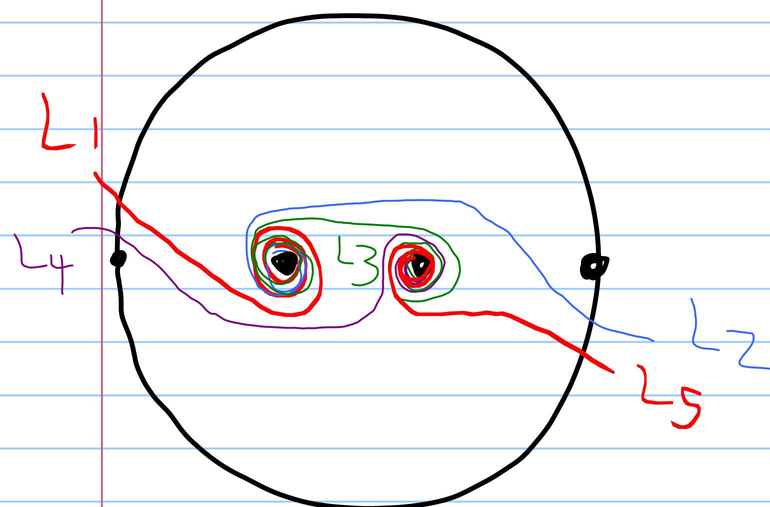
Putting these rules together:

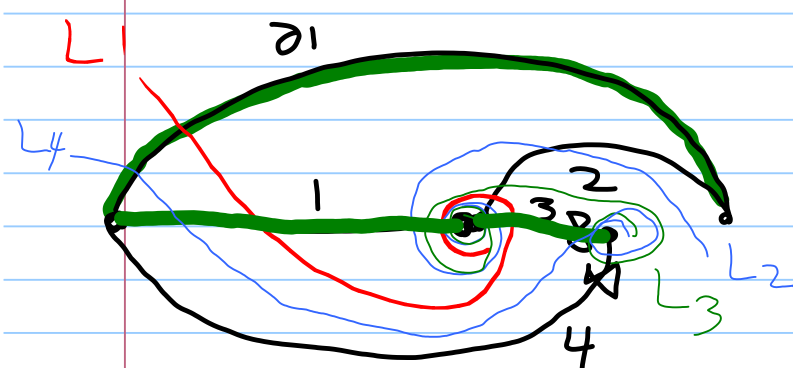


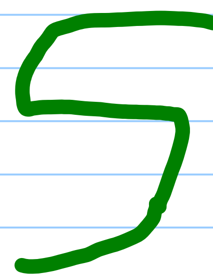
3 **Example:**



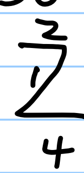
gives laminations







No lamination crosses τ_4 so crossing quadrilateral in negative formation impossible.



only L_1 crosses $\partial_1, \tau_1,$ and τ_3

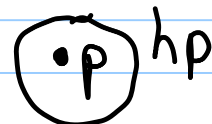
so column corresponding to τ_1 is indeed

$$\begin{array}{ccccc} L_1 & L_2 & L_3 & L_4 & L_5 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

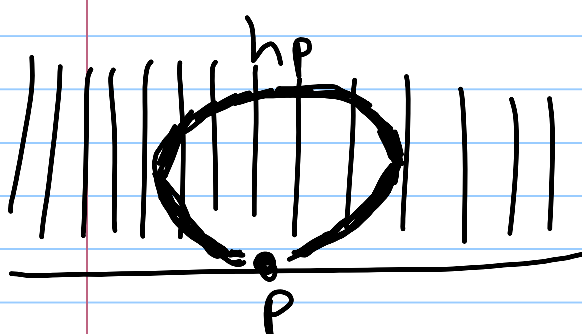
④ Similarly, all five rows/cols give Is.

We now switch gears and talk about Teichmüller interpretations of cluster variables / arcs.

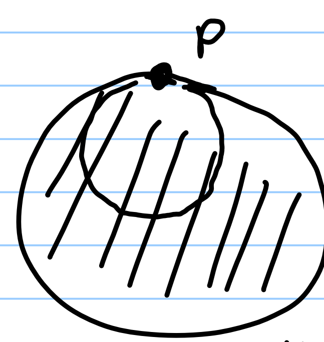
Def: A horocycle, at an ideal point p, is a set of points which are all equidistant to p.

Topologically: 

But in lift to hyperbolic upper half plane or the Poincare disk,



or



Def: The decorated Teichmüller space $\widetilde{\mathcal{T}}(S,M)$ is parametrized by data consisting of
- a point in $\mathcal{T}(S,M)$,

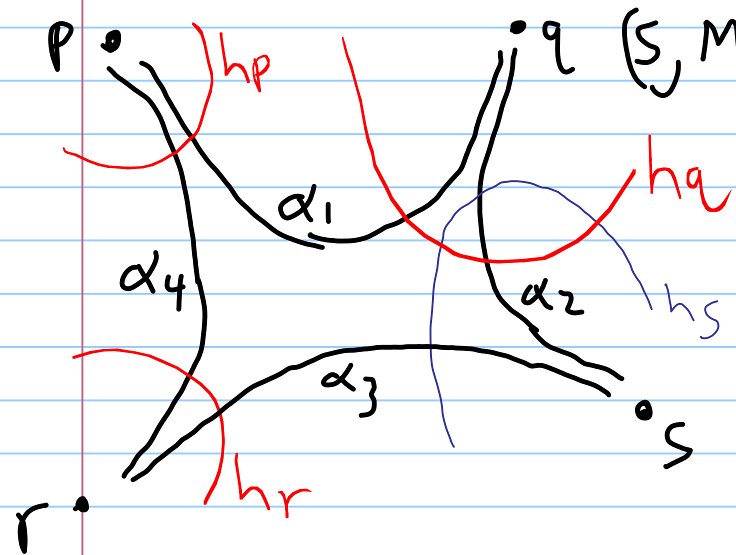
and
- a choice of horocycle around each cusp from M.

Def (Penner): For an arc E on (S,M) and a choice $\Sigma \in \widetilde{\mathcal{T}}(S,M)$, the length $\ell_\Sigma(E) :=$ length of the geodesic rep. of E

5 between intersections with horocycles.

If horocycles chosen largely enough so that they intersect, $\ell_\Sigma(\gamma)$ is negative instead.

In topological viewpoint



(S, M) $M = \{p, q, r, s\}$

In this e.g., $\ell_\Sigma(\alpha_2)$ is negative.

Def: The λ-length of E is defined as $\lambda_\Sigma(E) = e^{\ell_\Sigma(E)/2} \in \mathbb{R}_{>0}$.

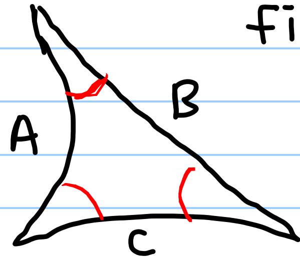
Penner coordinates for decorated Teichmüller space:

Thm (Penner): For any triangulation $T = \{E_i\}_{i=1}^n$ without self-folded triangles, the map

$$\prod_{\gamma \in T \cup \{\text{Boundary Arcs}\}} \lambda(\gamma) : \widetilde{\mathcal{T}}(S, M) \to \mathbb{R}_{>0}^{n+c}$$

is a homeomorphism.

6 **Sketch of Proof:** If our surface was a single triangle with fixed vertices, then choosing

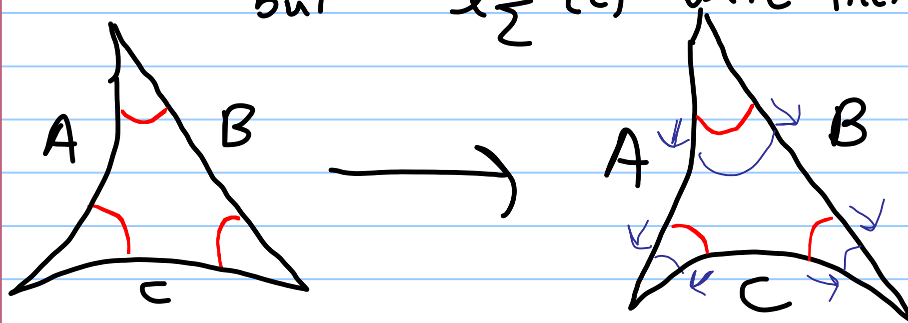


$\ell_\Sigma(A), \ell_\Sigma(B), \ell_\Sigma(C)$

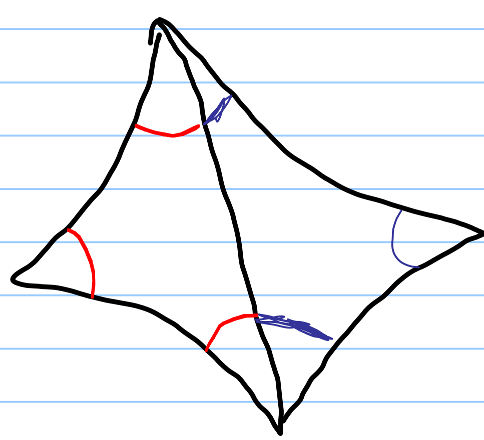
[Each $\ell_\Sigma(\cdot) \in (-\infty, \infty)$ so $\lambda_\Sigma(\cdot) \in \mathbb{R}_{>0}$]

uniquely determines the decoration with horocycles.

e.g. if $\ell_\Sigma(A), \ell_\Sigma(B)$ held fixed but $\ell_\Sigma(C)$ were increased,

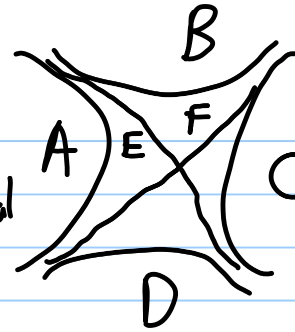


$\Rightarrow$ Lengths on all triangles determines decorated triangles and a unique way to glue adjacent triangles.



Also from λ-lengths (or $\ell_\Sigma(E)$) for all arcs of a triangulation + boundary, we can obtain lengths of any arc in surface.

⑦ Ptolemy Relation



For any ideal quadrilateral and $\Sigma \in \widetilde{\mathcal{T}}(S,M)$, we have

$$\lambda_\Sigma(E)\lambda_\Sigma(F) = \lambda_\Sigma(A)\lambda_\Sigma(C) + \lambda_\Sigma(B)\lambda_\Sigma(D).$$

Not just algebraic statement, but statement about exponentials of these hyperbolic lengths.

Notice that this is a "tropical"-like statement about lengths

$$e^{l_\Sigma(E)/2} e^{l_\Sigma(F)/2} = e^{l_\Sigma(A)/2} e^{l_\Sigma(C)/2} + e^{l_\Sigma(B)/2} e^{l_\Sigma(D)/2}$$

$$\Rightarrow l_\Sigma(E) + l_\Sigma(F) = \log\left(e^{l_\Sigma(A) + l_\Sigma(C)} + e^{l_\Sigma(B) + l_\Sigma(D)}\right)$$

Point: Let $x_{E_i} := \lambda_\Sigma(E_i)$ for each $E_i \in T$, a triangulation with no self-folded triangles plus boundary arcs.

Then, choice of $\{x_{E_i}\}$'s (as in $\mathbb{R}_{>0}^{n+c}$) uniquely determines data $\Sigma \in \widetilde{\mathcal{T}}(S,M)$

$\Rightarrow$ all other $\lambda_\Sigma(\gamma)$ for γ another arc of (S,M) determined.

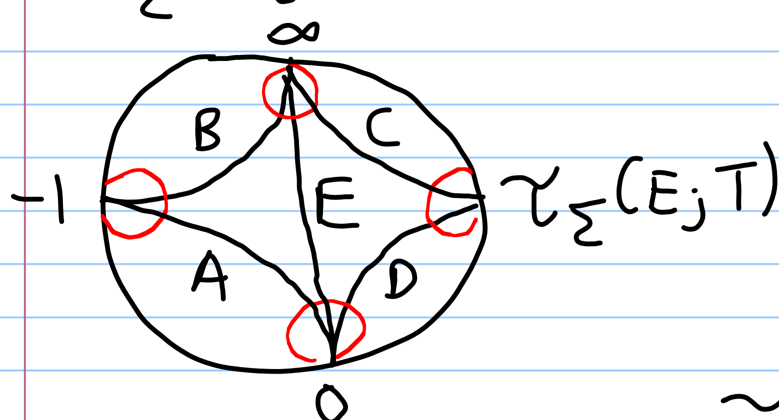
8. By iterations of Ptolemy Relations, each $\lambda_\Sigma(\gamma)$ is a Laurent polynomial in the x_{E_i}'s, and can be thought of as a function acting on points of $\tilde{\mathcal{T}}(S,M)$.

These are the cluster variables.

Relation to shear coordinates

Given a hyperbolic structure (undecorated) $\Sigma \in \mathcal{T}(S,M)$ and triangulation $T=\{E_i\}$

$\tau_\Sigma(E;T)$ = cross ratio

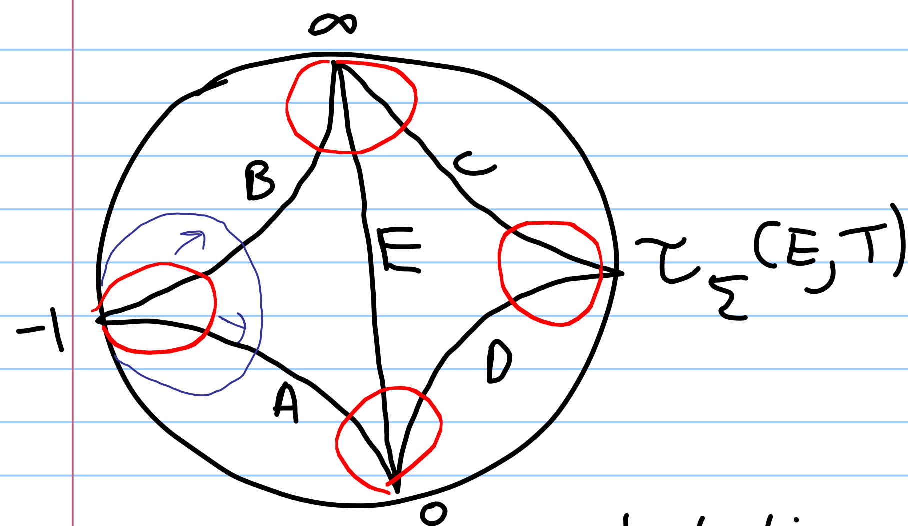


We can lift to an element $\tilde{\Sigma} \in \tilde{\mathcal{T}}(S,M)$ by choosing horocycles and then

$$\tau_\Sigma(E;T) = \frac{\lambda_{\tilde{\Sigma}}(A)\,\lambda_{\tilde{\Sigma}}(C)}{\lambda_{\tilde{\Sigma}}(B)\,\lambda_{\tilde{\Sigma}}(D)}$$

Does not depend on lift $\tilde{\Sigma}$, i.e. choice of horocycles.

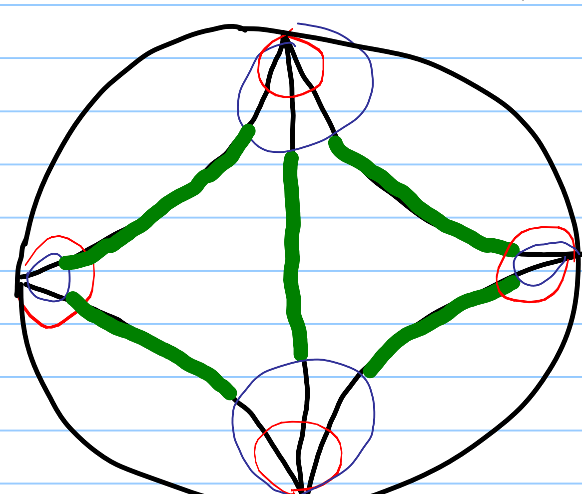
(9) Notice, if we make a horocycle bigger/smaller, it affects consecutive sides of quadilateral equally $\Rightarrow$ numerator and denominator equally.



Question came up last time:

Why do $\ell_{\widetilde{\Sigma}}(A), \ldots, \ell_{\widetilde{\Sigma}}(D)$ not determine $\ell_{\widetilde{\Sigma}}(E)$?

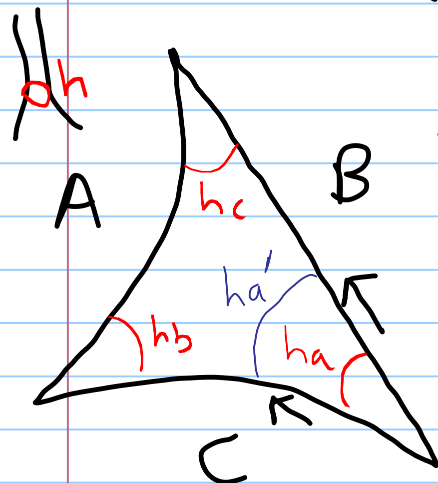
Answer: shrinking horocycles on left and right while expanding horocycles on top and bottom can keep $\ell_{\widetilde{\Sigma}}(A), \ldots, \ell_{\widetilde{\Sigma}}(D)$ fixed while shrinking $\ell_{\widetilde{\Sigma}}(E)$.



10 Some more Facts about λ-lengths

Note that since horocycles lie away from cusps, their hyperbolic lengths, or lengths between intersections with arcs of a triangulation, are well-defined.

We denote these as $L(h)$ or $L(h_c)$, etc.

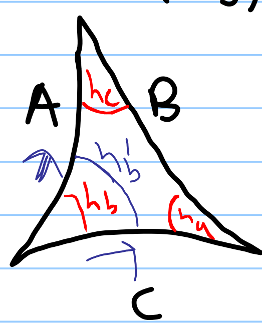


__Lemma__: In a __decorated__ triangle, we have the identity

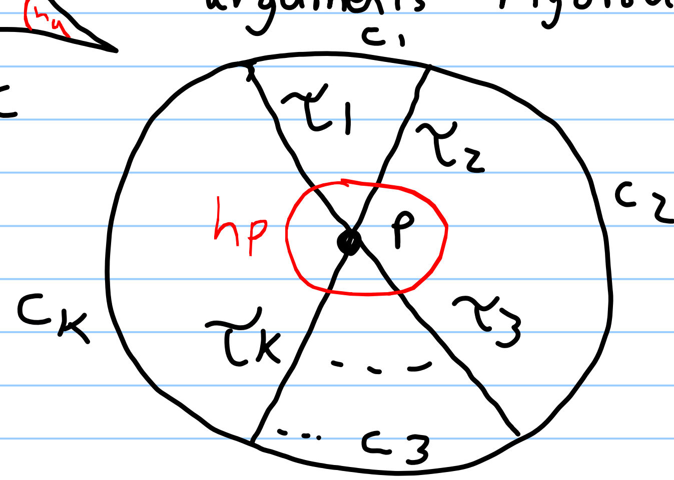
$$L(h_a) = \frac{\lambda_\Sigma(A)}{\lambda_\Sigma(B)\lambda_\Sigma(C)}$$

__Sketch of Proof__: as $L(h_a')$ gets bigger, $\lambda_\Sigma(B)$ and $\lambda_\Sigma(C)$ get smaller.

If $L(h_b)$ changed, $\lambda_\Sigma(A)$ and $\lambda_\Sigma(C)$ grow proportionally.



Integrating metric makes arguments rigorous.



Adding up contributions from triangles, we get

Cor: Around a puncture p incident to $\tau_1, \dots, \tau_K$ with opposite edges labeled as above,

$$L(h_p) = \sum_{i=1}^{K} \frac{\lambda_\Sigma(c_i)}{\lambda_\Sigma(\tau_i)\lambda_\Sigma(\tau_{i+1})}$$

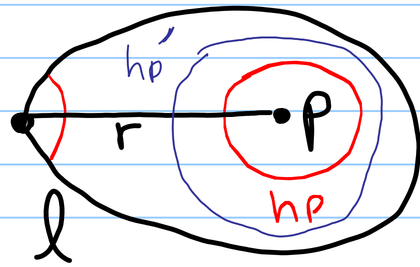
[letting τ_{K+1} denote τ_1]

$$= \frac{\sum_{i=1}^{k} \sigma^k\left(\lambda_\Sigma(c_1)\lambda_\Sigma(\tau_3)\cdots\lambda_\Sigma(\tau_k)\right)}{\lambda_\Sigma(\tau_1)\cdots\lambda_\Sigma(\tau_K)}$$

where σ is cycle permutation $(1\,2 \cdots K)$.

Def: Define two horocycles h and h' around p to be conjugate if $L(h') = 1/L(h)$.

Cor: In a once-punctured monogon ℓ with conjugate horocycles h_p and h_p', then



[$\Sigma' \in \mathcal{T}(S,M)$ has h_p' instead of h_p.]

$$L(h_p) = \frac{\lambda_\Sigma(\ell)}{\lambda_\Sigma(r)^2}$$

regardless of choice of $\Sigma \in \mathcal{T}(S,M)$

and

$$\lambda_{\Sigma'}(\ell) = \lambda_\Sigma(\ell) = \lambda_\Sigma(r)\cdot\lambda_{\Sigma'}(r)$$

(12) **Pf:** First statement follows from earlier Lemma.

For second, notice $\lambda_{\Sigma'}(\ell)=\lambda_{\Sigma}(\ell)$ as ℓ does not intersect h_p nor h_p'.

Since h_p' defined such that

$L(h_p') = 1/L(h_p)$, we have

$$L(h_p') = \lambda_{\Sigma'}(\ell)/\lambda_{\Sigma'}(r)^2 \quad \&$$

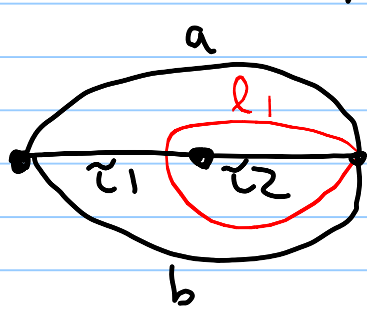
$$L(h_p') = \lambda_{\Sigma}(r)^2/\lambda_{\Sigma}(\ell), \Rightarrow$$

$$\lambda_{\Sigma'}(\ell)\,\lambda_{\Sigma}(\ell) = \lambda_{\Sigma}(r)^2\lambda_{\Sigma'}(r)^2.$$

$$\begin{array}{c} \lambda_{\Sigma'}(\ell)\,\lambda_{\Sigma}(\ell) \\ \| \\ \lambda_{\Sigma}(\ell)^2 \\ \| \\ \lambda_{\Sigma'}(\ell)^2 \end{array}$$

Taking square-roots of both sides finishes the proof.

Moral: Combinatorially, if we have a bi-gon



By Ptolemy Relation

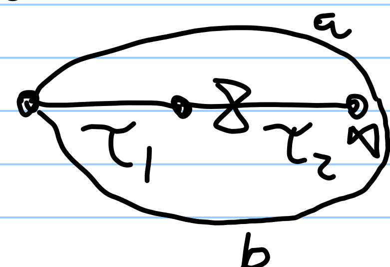
$$\ell_{\Sigma}(\tau_1)\,\ell_{\Sigma}(\ell_1) = \left(\ell_{\Sigma}(a) + \ell_{\Sigma}(b)\right)\ell_{\Sigma}(\tau_2)$$

Dividing both sides by $\ell_{\Sigma}(\tau_2)$,

(13) <u>we now get</u>

$$\ell_{\Sigma}(\tau_1)\,\ell_{\Sigma'}(\tau_2) = \ell_{\Sigma}(a) + \ell_{\Sigma}(b)$$

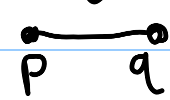
$\Rightarrow \ell_{\Sigma'}(\tau_2)$ behaves like tagged arc



This allowed <u>Fomin-Thurston</u> to think of cluster variables corr. to <u>tagged arcs</u> as

"Take $\Sigma \in \tilde{\mathcal{T}}(S, M)$,

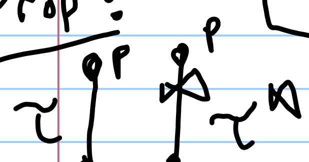
if τ notched at p or q,



change decoration Σ by changing horocycle at p and/or q to its <u>conjugate</u>. (Σ').

Then $\boxed{x_{\tau} = \lambda_{\Sigma'}(\tau)}$.

<u>Prop:</u>



$$\boxed{x_{\tau^{\bowtie}} = L(h_p)\cdot\lambda_{\Sigma}(\tau)}$$