Lecture 8: The Laurent Phenomenon
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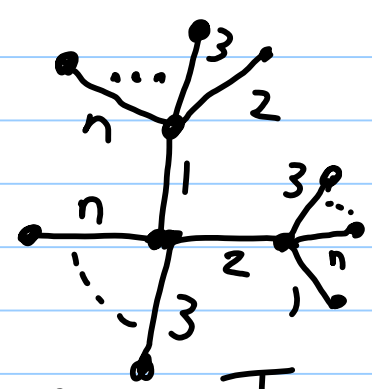
Note Title 2/13/2011

① Today we prove the <u>Laurent Phenomenon</u> (Fomin-Zelevisky 2001) and give some applications.

While the main application of the Laurent Phenomenon is to cluster algebras, the result actually applies more generally. We describe the statement in this context.

Some notation before the <u>Caterpillar Lemma</u>:

Fix an integer $n \geq 2$, and let T be the infinite degree n tree and we label edges between t & t' using the set $\{1, 2, \ldots, n\}$ so that each of the n edges incident to t get a different label.



Let $\mathbb{A}$ be a unique factorization domain (e.g. $\mathbb{Z}$, $\mathbb{Q}[c_1, \ldots, c_r]$).

For each edge $\overset{k}{t \text{ ——— } t'}$ in T,

we associate a polynomial P in $\mathbb{A}[x_1, \ldots, x_n]$ s.t. x_k does not divide P. We call P an <u>exchange polynomial</u>.

(2) The collection of all of these polynomials is known as a <u>generalized exchange pattern.</u>

Pick a root t_0 of T and we define an initial cluster to be $\{x_1(t_0), x_2(t_0), \dots, x_n(t_0)\}$.

we obtain $x_i(t)$ for $i \in \{1, 2, \dots, n\}$, $t \in T$ by iterating the exchanges

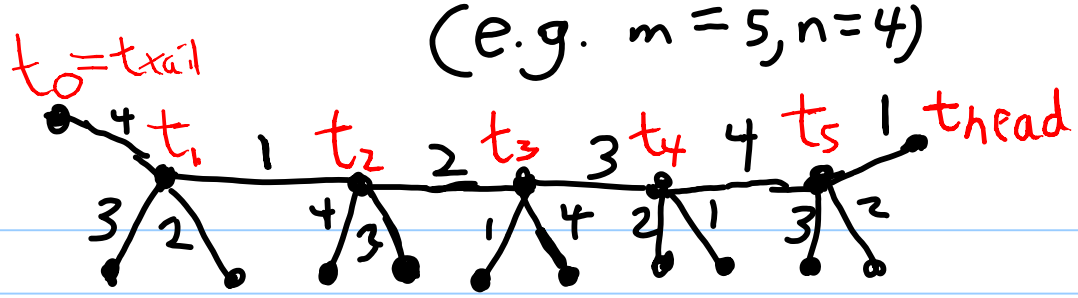
$$\underset{t}{\bullet} \overset{P}{\underset{k}{\text{———}}} \underset{t'}{\bullet}$$

$$x_i(t') := \begin{cases} x_i(t) & \text{if } i \neq k \\ \dfrac{P(x_1(t), \dots, x_n(t))}{x_k(t)} & \text{if } i = k. \end{cases}$$

Next, for $m \geq 1$, let $T_{m,n}$ denote a <u>Caterpillar tree</u> with m vertices $(t_1, t_2, \dots, t_m)$ on the spine, each with degree n and

$m(n-2)+2$ leaf vertices.

③ (e.g. $m=5, n=4$)



Thm (Caterpillar Lemma)

Assume that a generalized exchange pattern on $T_{m,n}$ satisfies the following extra conditions:

i) For any edge $t \overset{P}{\underset{k}{\text{———}}} t'$,

Polynomial P does not depend on x_k and is not divisible by x_i for any $i \in \{1, 2, \dots, n\}$.

ii) For any pair of edges $\underset{i}{\bullet}\overset{P}{\text{———}}\bullet\overset{Q}{\text{———}}\underset{j}{\bullet}$,

the polynomials P and $Q_0 := Q|_{x_i=0}$ are coprime in $\mathbb{A}[x_1, \dots, x_n]$.

iii) For any three edges $\underset{i}{\bullet}\overset{P}{\text{———}}\bullet\overset{Q}{\text{———}}\underset{j}{\bullet}\overset{R}{\text{———}}\underset{i}{\bullet}$

there exists a nonnegative integer b, and Laurent monomial L coprime to P s.t.

$$L \cdot Q_0^b \cdot P = R|_{x_j \leftarrow \frac{Q_0}{x_j}}.$$

[replace x_j w/ Q_0/x_j in R]

Then, each element $x_i(t)$ $\left(\begin{matrix} i \in \{1, 2, \dots, n\} \\ t \in T_{m,n} \end{matrix}\right)$ is a Laurent polynomial in $x_1(t_0), \dots, x_n(t_0)$ with coefficients in $\mathbb{A}$.

4 **Remark**: As we will see later, if all the polynomials in the generalized exchange pattern are binomials then the restrictions (i) & (iii) above agree with the axioms of a cluster algebra.

To prove the above Theorem, we will use the notation

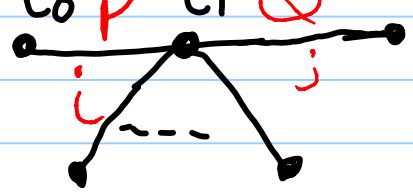
$$\mathcal{L}(t) := \mathbb{A}[x_1(t)^{\pm 1}, \dots, x_n(t)^{\pm 1}].$$

Goal: show that each $x_i(t)$ is in $\mathcal{L}(t_0)$.

Since $\mathcal{L}(t_0)$ is a unique factorization domain, we can discuss gcd's (well-defined up to units, Laurent monomials in $x_1(t_0), \dots, x_n(t_0)$ with coefficients in $\mathbb{A}^\times$.)

To prove $\{x_1(t), \dots, x_n(t)\} \subset \mathcal{L}(t_0)$, we use induction on m (the size of $T_{m,n}$'s spine).

If $m=1$: t_0 p t_1 Q $t_2 = t_{head}$



[t_1 of degree n]

(5) $x_k(t_2) = x_k(t_1) = x_k(t_0)$ if $k \neq i, j$,

$$x_i(t_2) = x_i(t_1) = \frac{P(x_1(t_0), \dots, x_n(t_0))}{x_i(t_0)}$$

$x_j(t_1) = x_j(t_0)$,

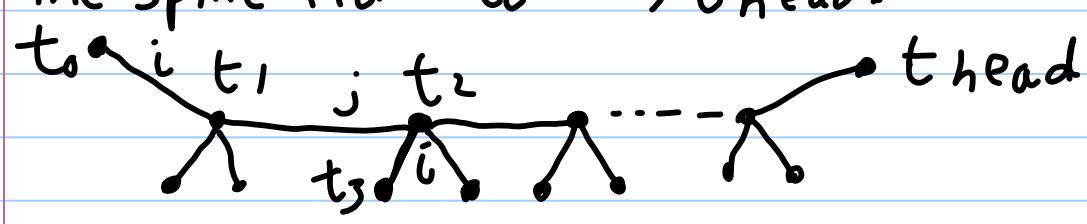
$$x_j(t_2) = \frac{Q\left(x_1(t_0), \dots, \frac{P}{x_i(t_0)}, \dots, x_n(t_0)\right)}{x_j(t_0)}$$

$\Rightarrow$ all $x_k(t)$'s $\in \mathcal{L}(t_0)$, for $t \in T_{n,1}$.

Assume $m \geq 2$ and that Theorem holds for shorter Caterpillars.

Suffices to show $x_i(t_{head}) \in \mathcal{L}(t_0)$ for $i \in \{1, 2, \dots, n\}$.

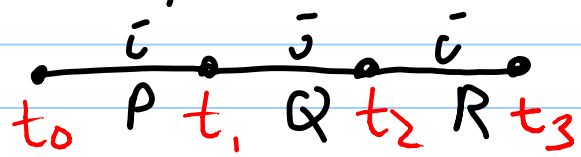
Let i & j $(i \neq j)$ denote the edge labels on the first two edges along the spine from $t_0 \longrightarrow t_{head}$.



⑥ Let t_3 (possibly a leaf) be the vertex adjacent to t_2 so that $t_2 \overset{i}{\text{———}} t_3$.

<u>Lemma:</u> The clusters $\underline{x}(t_1) := \{x_1(t_1), \dots, x_n(t_1)\}$, $\underline{x}(t_2)$, $\underline{x}(t_3) \subset \mathcal{L}(t_0)$. Furthermore $\gcd(x_i(t_3), x_i(t_1)) = 1$, as does $\gcd(x_j(t_2), x_i(t_1)) = 1$.

<u>Pf of Lemma:</u>



As we saw from looking at the $m=1$ case, all elements in these three clusters are clearly in $\mathcal{L}(t_0)$, except for $x_i(t_3)$.

Thinking of P, Q, R as polynomials only in the appropriate variables $x_i(t_1), x_j(t_1)$. Furthermore P and R do not depend on x_i while Q does not depend on $x_j(t_1)$. Thus we can denote P, Q, R as univariate polynomials.

Let $x = x_i(t_0)$, $y = x_j(t_0) = x_j(t_1)$,

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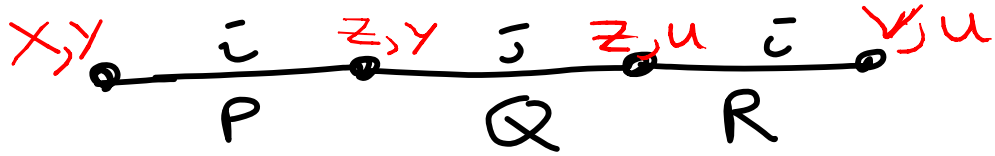

$$z = \frac{P(y)}{x} = x_i(t_1), \quad u = \frac{Q(z)}{y} = \frac{Q\left(\frac{P(y)}{x}\right)}{y} = x_j(t_2),$$

$$v = \frac{R(u)}{z} = \frac{R\left(\frac{Q(z)}{y}\right)}{z} = \frac{R\left(\frac{Q\left(\frac{P(y)}{x}\right)}{y}\right)}{z} = x_i(t_3).$$

We wish to show that $v \in \mathcal{L}(t_0)$, so we rewrite v as

$$v = \left[\frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z}\right] + \frac{R\left(\frac{Q(0)}{y}\right)}{z}.$$

Note that $R\left(\frac{Q(0)}{y}\right) = R\Big|_{x_j \leftarrow \frac{Q_0}{x_i}}$

(x_j = shorthand for $x_j(t_0) = x_j(t_1)$)

$\left(Q_0 = Q\big|_{x_i(t_1)=0}\right)$

Note that R does not depend on $x_i(t_2) = z$ so the term $R\left(\frac{Q(z)}{y}\right)$ only contains z's coming from $Q(z)$. Consequently,

(8) $R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)$ must be divisible by z & so the first term is in $\mathcal{L}(t_0) = \mathbb{A}[x^{\pm}, y^{\pm}, \dots]$

Second term is in $\mathcal{L}(t_0)$ since by condition (iii), $R\left(\frac{Q(0)}{y}\right) = L \cdot Q_0^b \cdot P$ where L is a Laurent monomial in $x(t_2)$ coprime to P, b is a nonnegative integer.

$$\frac{L \cdot Q_0^b \cdot P(y)}{z} = L \cdot Q_0^b(x) \quad (xz = P(y))$$

$\Rightarrow v \in \mathcal{L}(t_0)$.

To see $\gcd(z, u) = 1$, we use

$$u = \frac{Q(z)}{y} \equiv \frac{Q(0)}{y} \mod z.$$

x, y invertible in $\mathcal{L}(t_0) \Rightarrow$

$$\gcd(z, u) = \gcd\left(z, \frac{Q(0)}{y}\right) = \gcd\left(\frac{P(y)}{x}, \frac{Q(0)}{y}\right)$$

$= \gcd(P(y), Q(0))$. Now, by condition (ii), $\gcd(P(y), Q(0)) = 1$.

(9) To see $\gcd(z, v) = 1$, we let $f(z) = R\left(\frac{Q(z)}{y}\right)$. $\Rightarrow$ we can write

$$v = \frac{f(z) - f(0)}{z} + L(y)Q(0)^b x.$$

Modulo z, $\lim_{z \to 0} \frac{f(z) - f(0)}{z} = f'(0)$, &

$$f'(0) = R'\left(\frac{Q(0)}{y}\right)\frac{Q'(0)}{y}, \text{ so}$$

<u>Modulo z</u>, (*) $v \equiv R'\left(\frac{Q(0)}{y}\right)\frac{Q'(0)}{y} + L(y)Q(0)^b x$

In (*), RHS is, with respect to x, a degree one polynomial with coefficients that are variables in $\underline{x}(t_0)$.

so (*) $\Rightarrow$ $v \equiv S_1(y) + L(y)Q(0)^b x \mod z$

where $S_1(y) = R'\left(\frac{Q(0)}{y}\right)\frac{Q'(0)}{y}$ $\left[z = \frac{P(y)}{x}\right]$

Since we want to show $\gcd(z, v) = 1$, we note that if d divides z and d divides v, then

(10) $d = d(y)$ is a polynomial in y that does not depend on x.

[This follows from $v \equiv S_1(y) + L(y)Q(0)^b x \mod z$]

not dependent on x

So $d(y)$ must divide $S_1(y)$, $L(y)Q(0)^b$, & $P(y)$.

<u>Claim:</u> $\gcd(L(y)Q(0)^b, P(y)) = 1$

Follows from (ii) $[\gcd(P(y), Q(0)) = 1]$ and (iii) $[\gcd(L(y), P(y)) = 1]$.

$\Rightarrow d(y) = 1$ and we have $\gcd(z, v) = 1$.

Lemma now proved. ////

We now turn our attention back to the main Thm, the

Caterpillar Lemma.

(11) Must show that given gen. exchange pattern satisfying (i), (ii), and (iii) satisfies

$x_k(t_{head}) \in \mathcal{L}(t_0)$ for all $k \in \{1, 2, \dots, n\}$.

By induction, t_1 and t_3 are closer to t_{head} than t_0 so

$x_k(t_{head}) \in \mathcal{L}(t_1)$ and $\mathcal{L}(t_3)$.

$\Rightarrow x_k(t_{head})$ is a Laurent poly in $\{x_1(t_0), \dots, x_n(t_0)\} \cup \{x_i(t_1)\}$ as well as $\{x_1(t_0), \dots, x_n(t_0)\} \cup \{x_i(t_3), x_j(t_2)\}$

Thus we can write

$$x_k(t_{head}) = \frac{f}{x_i(t_1)^a} = \frac{g}{x_j(t_2)^b x_i(t_3)^c}$$

where $f, g \in \mathcal{L}(t_0)$, $a, b, c \in \mathbb{Z}_{\geq 0}$.

Note that if $x_k(t_{head})$ contains a Laurent monomial with a positive power of $x_i(t_1) = \frac{P}{x_i(t_0)}$, $x_j(t_2) = \frac{Q\left(\frac{P}{x_i(t_0)}\right)}{x_j(t_0)}$

(12) and $x_i(t_3) \in \mathcal{L}(t_0)$ by Lemma is still in $\mathcal{L}(t_0)$.

Hence why we are reduced to considering negative powers.

However Lemma also tells us that $\gcd(x_i(t_1), x_j(t_2))$
$= \gcd(x_i(t_1), x_i(t_3)) = 1$

and so only way $x_k(t_{head})$ can be written in both ways is if $x_k(t_{head}) \in \mathcal{L}(t_0)$.

Caterpillar Lemma now proven. ▨

Application: Proving Laurentness for a one-dimensional recurrence.

Let $\{x_0, x_1, x_2, \dots\}$ be a sequence defined by $x_{m+n} = \dfrac{F(x_{m+1}, \dots, x_{m+n-1})}{x_n}$

$(F \in \mathbb{A}[x_1, \dots, x_{n-1}])$

where F is a polynomial in $n-1$ variables that does not depend on m.