Lecture 9: Somos Sequences (2-16-11)

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Note Title 2/14/2011

① Recall that last time, we proved the Laurent Phenomenon, if a generalized exchange pattern on $T_{m,n}$ satisfies

i) For any $\overset{t}{\bullet}\!-\!\overset{P}{\underset{k}{\bullet}}\!-\!\overset{t'}{\bullet}$, Poly P does not depend on x_k and is not divisible by any x_i, $i \in \{1,2,\dots,n\}$,

ii) For any $\overset{P}{\underset{i}{\bullet}}\!-\!\overset{Q}{\underset{j}{\bullet}}$ P and $Q_0 := Q|_{x_i=0}$ are coprime in $\mathcal{A}[x_1,\dots,x_n]$.

iii) For $\overset{P}{\underset{i}{\bullet}}\!-\!\overset{Q}{\underset{j}{\bullet}}\!-\!\overset{R}{\underset{i}{\bullet}}$, there exists a nonnegative integer b and Laurent monomial L, coprime to P,

s.t. $L \cdot Q_0^b \cdot P = R|_{x_j \leftarrow \frac{Q_0}{x_j}}$.

[replace x_j w/ Q_0/x_j in R]

Then, each element $x_i(t)$ $\left(\begin{smallmatrix} i \in \{1,2,\dots,n\} \\ t \in T_{m,n} \end{smallmatrix}\right)$ is a Laurent polynomial in $x_1(t_0),\dots,x_n(t_0)$ with coefficients in $\mathcal{A}$.

Application: Proving Laurentness for a one-dimensional recurrence.

Let $\{x_0, x_1, x_2, \dots\}$ be a sequence defined by $x_{m+n} = \dfrac{F(x_{m+1},\dots,x_{m+n-1})}{x_n}$

($F \in \mathcal{A}[x_1,\dots,x_{n-1}]$)

where F is a polynomial in $n-1$ variables that does not depend on m.

(2) Thm Let F be as above and further satisfying

(a) F is not divisible by x_i for $i \in \{1, 2, \ldots, n\}$.

For $m \in \{1, 2, \ldots, n-1\}$, let

$$Q_m := F(x_{m+1}, \ldots, x_{n-1}, 0, x_1, \ldots, x_{m-1})$$

(b) Each Q_m is an irreducible element of $\mathbb{A}[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$.

and there is one last requirement (c), that takes more notation to state:

we define a sequence of auxiliary polynomials $G_{n-1}, G_{n-2}, \ldots, G_1, G_0$ in $\mathbb{A}[x_1, \ldots, x_{n-1}]$.

$G_{n-1} := F(x_1, \ldots x_{n-1})$, and for $1 \le K \le n-1$

$$\widetilde{G}_{K-1} := G_K\left(x_1, \ldots, x_{K-1}, \frac{Q_K}{x_K}, x_{K+1}, \ldots, x_{n-1}\right).$$

$\widetilde{G}_{K-1}$ is an element of $\mathbb{A}[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$

So we let L_{K-1} be a Laurent monomial in $x_1, \ldots, x_{n-1}$ with coefficients in $\mathbb{A}$ s.t. $\widetilde{G}_{K-1} := \widetilde{G}_{K-1} / L$ is

(3) a polynomial in $A[x_1, \ldots, x_{n-1}]$ that is not divisible by any of the x_i or a non-invertible scalar in A.

We then define $G_{K-1} := \dfrac{\widetilde{G}_{K-1}}{Q_0^b}$ where b is a nonnegative integer, the maximal power of Q_0 dividing $\widetilde{G}_{K-1}$.

The last condition is

(c) $G_0 = F$. [after iterating this procedure.]

If (a), (b), (c) satisfied, all x_m's are Laurent.

Example (Somos-4)

$$x_{n+4} = \frac{x_{n+1}\, x_{n+3} + x_{n+2}^2}{x_n}.$$

e.g. if $x_1 = x_2 = x_3 = x_4 = 1$,

$\{x_n\} = \{1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, \ldots\}$

$F = x_1 x_3 + x_2^2 = G_3$

$Q_3 = F(0, x_1, x_2) = x_1^2$

$\widetilde{G}_2 = x_1\left(\frac{x_1^2}{x_3}\right) + x_2^2 \Rightarrow \widetilde{G}_2 = x_1^3 + x_2^2 x_3 = G_2$

(4) $Q_2 = F(x_3, 0, x_1) = \underline{x_1 x_3}$

$\widetilde{G_1} = x_1^3 + \left(\frac{x_1 x_3}{x_2}\right)^2 x_3 \Rightarrow \widetilde{\widetilde{G_1}} = x_1 x_2^2 + x_3^3 = G_1$

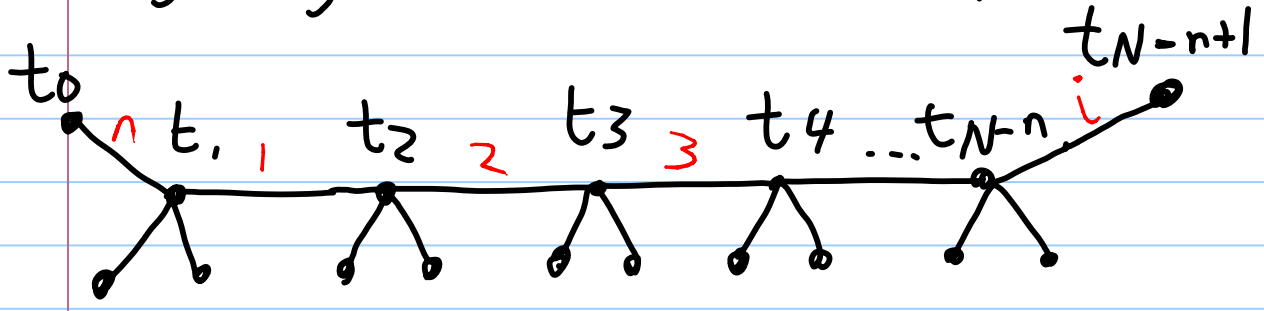
$Q_1 = F(x_2, x_3, 0) = \underline{x_3^2}$

$\widetilde{G_0} = \left(\frac{x_3^2}{x_1}\right) x_2^2 + x_3^3 \Rightarrow \widetilde{\widetilde{G_0}} = x_2^2 + x_1 x_3 = G_0$

Since the Q_i's are irreducible and $G_0 = F$, we conclude that all x_n in Somos-4 are Laurent polynomials.

Pf of one-dim recurrence Thm:

To prove Laurentness of x_N for large N, we build a Caterpillar



$t_0 = \{x_1, x_2, \ldots, x_n\}$

$t_1 = \{x_{n+1}, x_2, \ldots, x_n\}$

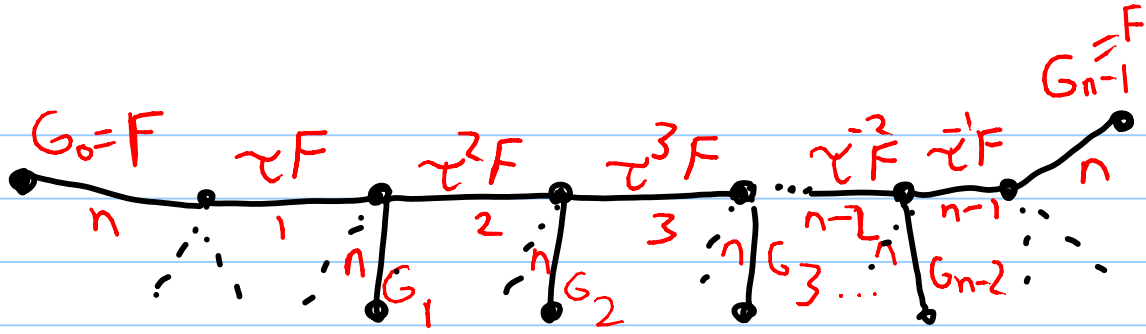
$t_2 = \{x_{n+1}, x_{n+2}, \ldots, x_n\}$

$\vdots$

Exchange F $\Rightarrow$ Exch. polys along spine

We attach Polynomials along the legs by using $G_{n-2}, \ldots, G_1$

(5)



where τ is a cyclic rotation of indices of F and rest of the legs of the caterpillar also assigned polys by cyclically rotating indices of G_k's.

Consequently, we can periodically extend this caterpillar all the way to t_{N+n-1}, and the axioms of a

a generalized exchange pattern are satisfied and by the Caterpillar Lemma, x_N is a Laurent polynomial $\{x_1, \ldots, x_n\}$.

Remark: In the $n=2$ case $x_m x_{m-2} = P(x_{m-1})$ where $P(0) \neq 0$, Caterpillar Lemma has a converse.

In argument due to David Speyer all x_n are Laurent polynomials in $\mathbb{A}[x_1^{\pm 1}, x_2^{\pm 1}]$ $\iff$ $\exists\, c \in \mathbb{A}$ s.t.

$$P(t) = c \cdot t^{\deg(P)} \cdot P\left(\frac{P(0)}{t}\right).$$

(6) If all x_n are Laurent,

then $x_3 = \frac{p(x_2)}{x_1}$, $x_4 = \frac{p\left(\frac{p(x_2)}{x_1}\right)}{x_2}$,

$x_5 = p\left(\frac{p\left(\frac{p(x_2)}{x_1}\right)}{x_2}\right) \Big/ \frac{p(x_2)}{x_1}$ Laurent

$\Rightarrow x_1 \cdot p\left(\frac{p\left(\frac{p(x_2)}{x_1}\right)}{x_2}\right) \equiv 0 \mod p(x_2)$

in $\mathbb{A}[x_1^{\pm 1}, x_2^{\pm 1}]$.

$\Rightarrow x_1 \cdot p\left(\frac{p(0)}{x_2}\right) \equiv 0 \mod p(x_2)$.

x_1, x_2 are units in $\mathbb{A}[x_1^{\pm 1}, x_2^{\pm 1}]$,

$\Rightarrow p(x_2) \mid x_1^{k_1} x_2^{k_2} \cdot p\left(\frac{p(0)}{x_2}\right)$

variable x does not appear in $p(x_2)$
so can let $k_1 = 0$ and $k_2 = \deg(p)$.

So $c' \cdot p(x_2) = x_2^{\deg p} \cdot p\left(\frac{p(0)}{x_2}\right)$

(c' must be nonzero constant) $\Rightarrow$

$$p(t) = \frac{1}{c'} \cdot t^{\deg p} \cdot p\left(\frac{p(0)}{t}\right).$$

Other direction follows as above.

(7) Connection to Cluster algebras

Thm: All cluster variables are Laurent polynomials in the initial cluster.

Pf: Follows from similar but variant of Caterpillar Lemma except that restriction (ii), $\gcd(P, Q_0) = 1$, is lifted and polynomials must be binomials.

Furthermore in the proof, when we show/need $\gcd(z, u) = 1$ and $\gcd(z, v) = 1$,

these statements are now thought of as in $\mathcal{L}(t_0) = \mathbb{A}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

as opposed to in the polynomial ring

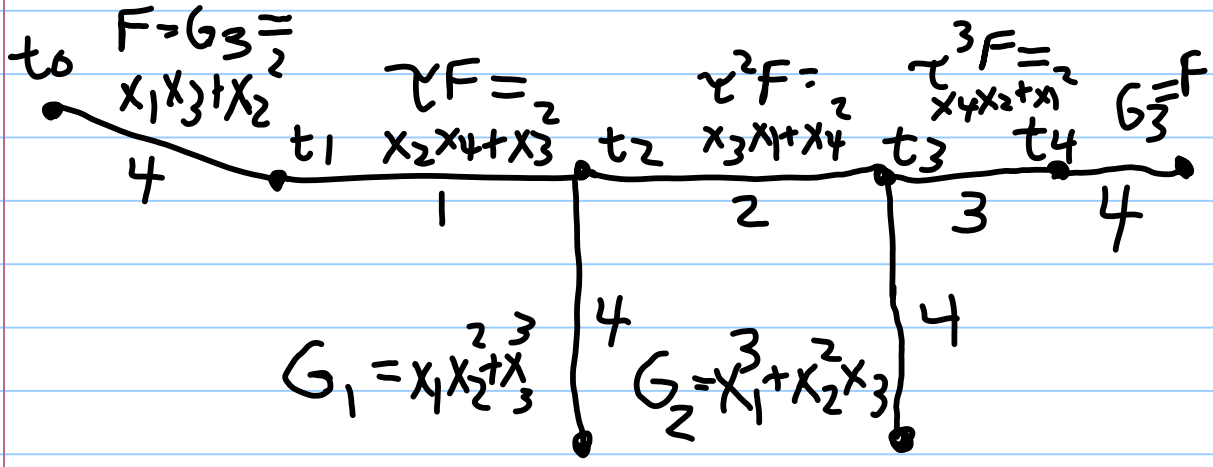
$\mathbb{A}[x_1(t_0), \ldots, x_n(t_0)]$.

Also note that we can use Caterpillar Lemma and cond. (iii) about edges

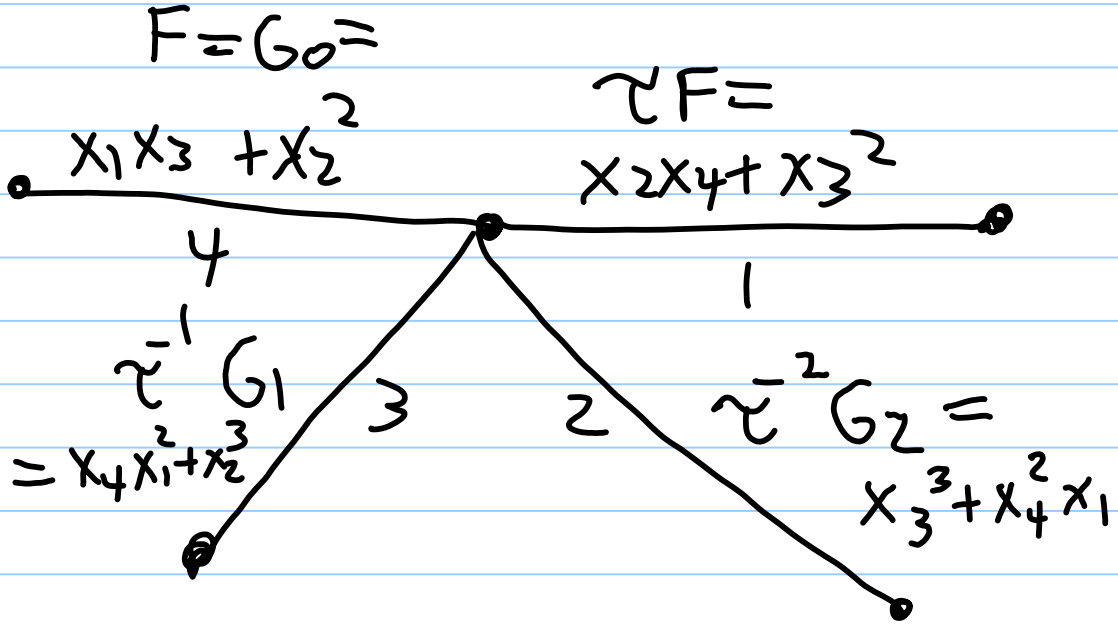
to reconstruct the initial exchange pattern for the corresponding Cluster algebra.

(8) We demonstrate this in the case of a one-dimensional recurrence where the exchange pattern corresp. to each t_i along the spine is just a cyclic rotation of the one before.

e.g. Somos-4



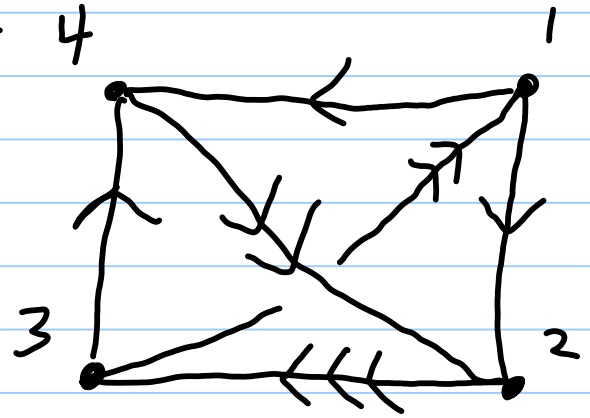
Thus we can construct pattern around t_1 as



(9) Thus consider cluster Aly

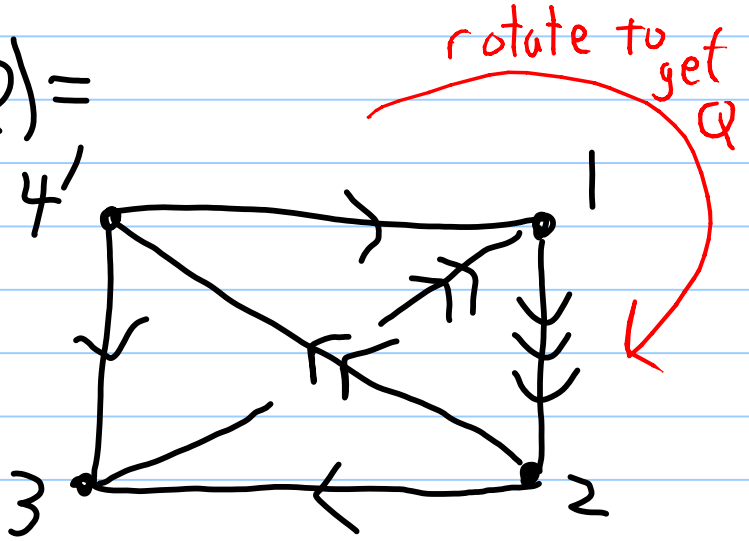
$$\mathcal{A}\left(\{x_1,x_2,x_3,x_4\}, \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 4 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & 1 & -2 \\ -2 & -1 & 0 & 3 \\ 1 & 2 & -3 & 0 \end{bmatrix} \\ B \end{array}\right)$$

As a quiver: $Q =$



if we mutate at v_4 or v_1 we obtain a rotation of Q:

e.g. $\mu_4(Q) =$



$\mu_1(Q)$ similar. Thus this cluster algebra will consistenty output exchange $x_{n+4}x_n = x_{n+1}x_{n+3} + x_{n+2}^2$.

(10) After proving Laurent Phenomenon, Fomin and Zelevinsky conjectured that expressions for cluster variables are also positive. (Positivity Conjecture)

We will talk about the pos. Conj. for "cluster algebras from surfaces" and other examples later in the course.

Example (Somos 4) $x_n x_{n-4} = x_{n-1}x_{n-3} + x_{n-2}^2$:

$$x_5 = \frac{x_2x_4 + x_3^2}{x_1}, \quad x_6 = \frac{x_2x_3x_4 + x_3^2 + x_1x_4^2}{x_1x_2},$$

$$x_7 = \frac{x_1x_2x_3x_4^2 + x_1x_3^3x_4 + x_1^2x_4^2 + x_2^3x_4^2 + 2x_2^2x_3^2x_4 + x_2x_3^4}{x_1^2x_2x_3}$$

$$x_8 = */x_1^3x_2^2x_3x_4, \quad x_9 = */x_1^3x_2^3x_3^2x_4$$

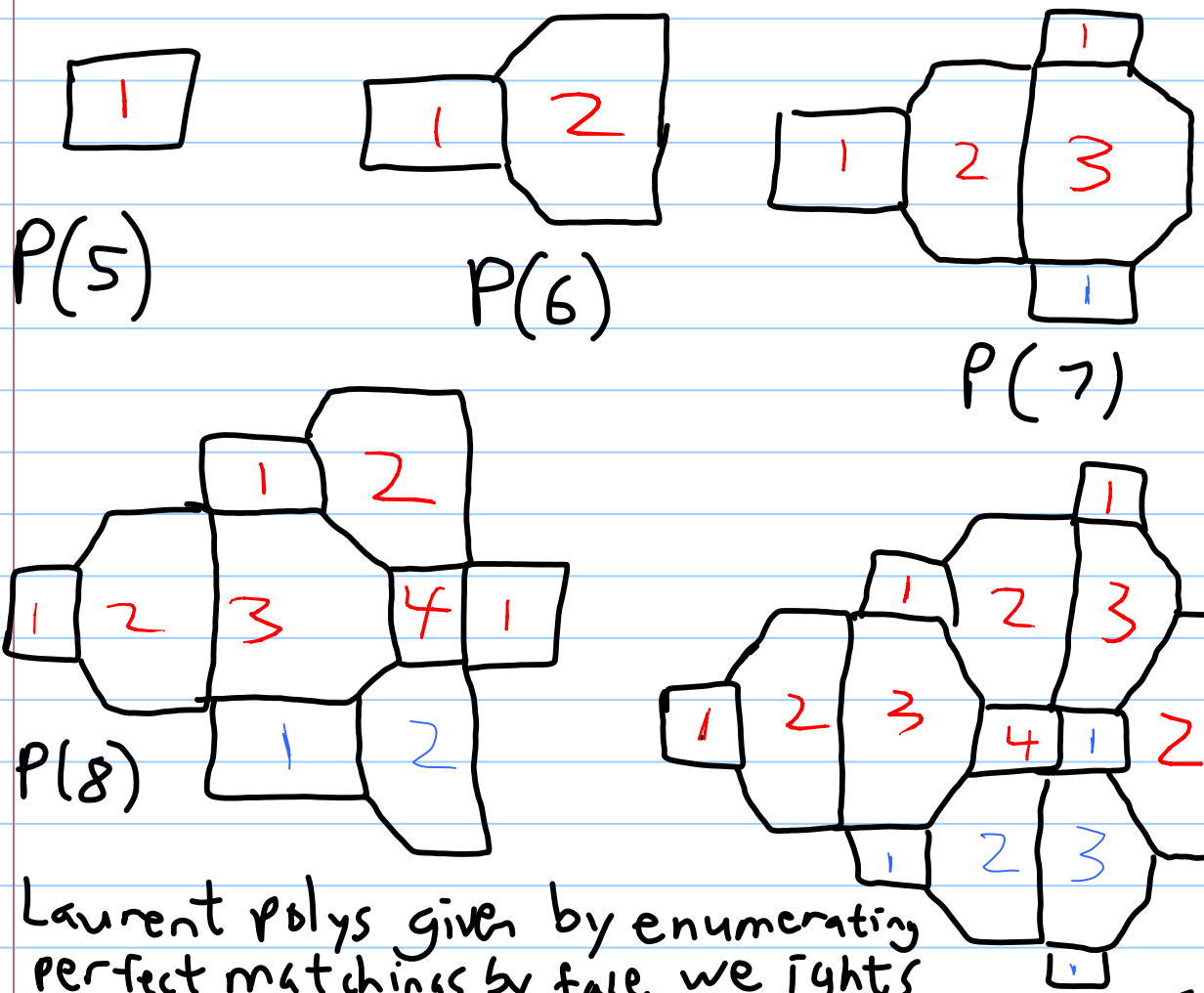
where $*$'s are positive polynomials.

Claim: All such x_n's are positive Laurent polynomials.

Integrality of Somos-4 even known before Caterpillar Lemma by several people in early 90's.

(11) A combinatorial proof of positivity of Somos-4 and other seqs appears in work of David Speyer in "perfect matchings in the octahedron recurrence".

Also proven by REACH group, led by Jim Propp in 2001-02 and related work by Bosquet-Mélou, Propp, and West, appearing on the arxiv in 2009: "perfect matchings for the three term Gale-Robinson sequences.

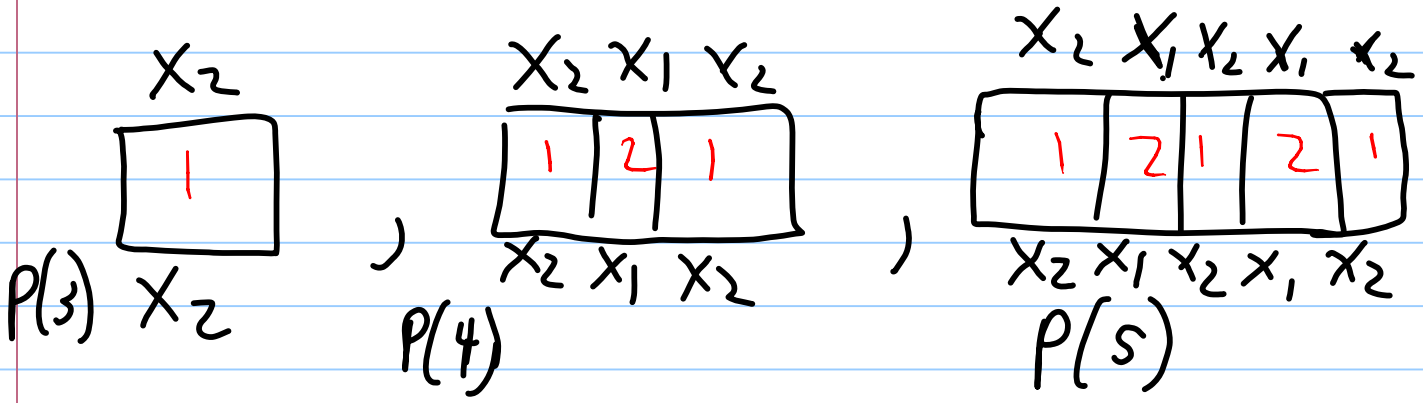


Laurent polys given by enumerating perfect matchings by tale weights

Speyer's general method uses "crosses and wrenches" read off from recurrence.

(12) Simpler case: $X_n X_{n-2} = X_{n-1}^2 + 1$

$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ Rank two affine case

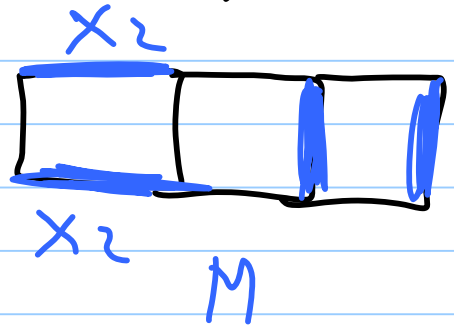


Cluster variables are positive Laurent polynomials in this case by

$$X_n = \sum_{\substack{M = \text{perfect} \\ \text{matching of} \\ P(n)}} \frac{X(M)}{X_1^{n-2} X_2^{n-3}}$$

where $X(M)$ uses edge weights

e.g.



$X(M) = X_2^2$

Special case of a Theorem for cluster algebras from surfaces that we will see later in the course.