Lecture 13: Kasteleyn Theory

In summary, in an effort to understand global symmetries (which induce automorphisms of the superpotential algebra), we study the cone \( N^+ \) corresponding to global symmetries with nonnegative weights.

\( N^+ \) is generated by perfect matchings of the dimer model corresponding to \( A \).

Today, we begin discussing techniques from algebraic combinatorics to enumerate perfect matchings of a dimer model (with weights).

We follow Sec 3 of Rick Kenyon's "Lectures on Dimers" arXiv: 0910.3129

Rem: Kenyon discuss dimer enumeration in the context of probability theory but is essentially combinatorics. We give the probabilistic language here accordingly.

Computing the number of dimer coverings of a bipartite planar graph using the Kasteleyn-Temperley-Fisher technique:

Def: Let \( G = (V, E) \) be a bipartite graph with a positive real function \( \omega: E \to \mathbb{R}_{>0} \) on edges,
We define a probability measure, called the Boltzmann measure, on dimer covers by

\[ M(G, w) \] has value \[ M(M) = \frac{\prod w(e)}{\sum_{M' \in \text{EM}} \prod w'(e)} \]

on a given dimer cover, i.e., perfect matching \( M \) of \( G \)

Note: \( \prod w(e) \) is also sometimes referred to as a Boltzmann weight of \( M \).

Furthermore, the denominator is often abbreviated as \( Z \) and called the partition function.

**Gauge equivalence:** Different weight functions \( w \) can easily give rise to the same measure \( M \).

For e.g., multiplying all \( w(e) \)'s for edges \( e \) incident to a fixed vertex \( v \) by a constant \( \lambda \) scales both the numerator and denominator by \( \lambda \) and leaves \( M \) unchanged.

**Def:** If \( w \) and \( w' \) are related to one another by a sequence of scalings at vertices as above, then we say weight functions \( w \) and \( w' \) are gauge-equivalent.

**Prop:** \( w \) and \( w' \) are gauge equivalent \( \iff \) for every face \( v_1v_2v_3 \) in cyclic order, the alternating products of Boltzmann weights, i.e.,
Let $w(e_1)w(e_3) \ldots w(e_{2k+1})$ and $w'(e_1)w'(e_3) \ldots w'(e_{2k+1})$ are equal. (In the case of a planar graph $G$.)

**PF:** We use techniques from graph homology and cohomology. Recall we already have seen coboundary map $d: \mathbb{Z}^Q_0 \to \mathbb{Z}^Q_1$ defined by $g \mapsto dg$ where

$$dg(v_1 \to v_2) = g(v_2) - g(v_1)$$

Today, we consider $d: \mathbb{R}^V \to \mathbb{R}^E$ for graph $G = (V,E)$ where we orient all edges of $E$ as $0 \to e_0$. We then define $f(0 \leftarrow e_0)$ as $-f(0 \to e_0)$ for $f \in \mathbb{R}^E$.

**Warning:** We are not considering $G$ to be dual to Quiver $(Q_0, A)$ today, but simply borrowing some terminology to analyze $G$.

**Defn:** We call $f \in \mathbb{R}^E$ satisfying $f(-e) = -f(e)$ a 1-form.

**Claim:** For a planar graph $G = (V,E)$, the set of 1-forms $F$ satisfying $df = 0$, $f_{H_{1F}}$ (here, $d: \mathbb{Z}_E \to \mathbb{Z}_F$ by $df$) called cocycles, is the same as the set $dF(\pi)$ of 1-forms satisfying $f \in \text{Im}(d: \mathbb{R}^V \to \mathbb{R}^E)$, called co-boundaries, of a face $\pi = (e_1, e_2, \ldots, e_k)$ in cyclic order.
Proof of Claim: This follows from standard holomorphy theory.

\[ \text{cocytes} = \ker d : \mathbb{R}^E \rightarrow \mathbb{R}^F \]
\[ \text{coboundaries} = \text{Im} d : \mathbb{R}^V \rightarrow \mathbb{R}^E \]

\[ H^1(G) = \frac{\ker d}{\text{Im} d} = \text{cocytes/coboundaries} \]

and when \( G \) is planar, after letting \( F^3 \)'s fill in \( G \) as \( 2 \)-cells, \( G \cong \text{disc or sphere } S^2 \) (if we infinite face).

Either way, \( H^1 = 0 \).

Remark: If \( G \) is not planar but is embedded on a genus \( g \) surface instead, then \( H^1(G) \cong \mathbb{R}^{2g} \).

For example, for a bipartite tiling on a torus, we get \( \ker d / \text{Im} d \cong \mathbb{R}^2 \).

We come back to this case later.

With this claim in hand, and a choice of weight function \( w : E \rightarrow \mathbb{R} \), consider the function \( \log w \) as a 1-form. i.e.,

\[ \log w(0 \leftarrow e \rightarrow 0) = -\log(0 \leftarrow e \rightarrow 0) \]

By definition of gauge-equivalence, \( w \sim w' \) if \( \exists f \in \mathbb{R}^V \) s.t.

\[ w'(e) = w(e) \cdot f(v'_1) f(v'_2) \]

for edge \( e = v_1 \leftarrow v_2 \).

By tweaking function \( f \), we can also incorporate orientations \( 0 \rightarrow \rightarrow \rightarrow \) and say \( w'(e) = \frac{w(e) \cdot f(v_2)}{f(v_1)} \) for some \( f \in \mathbb{R}^V \).
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Equivalently, \( \log w' - \log w = dF \) for some \( F \in \mathbb{R}^V \).

Thus we need \( \log w' - \log w \) is a coboundary\(^{(1)}\), (in the planar graph case) that

\[ \log w' - \log w \text{ is a cocycle} \]

\( \Rightarrow \) \( d(\log w' - \log w) = 0 \Rightarrow d \log w' = d \log w \)

\( \Rightarrow \) \( \forall \text{ Faces} \)

\[
\begin{array}{c}
\log w(e_1) - \log w(e_2) + \cdots + \log w(e_{k-1}) - \log w(e_k) \\
= \log w'(e_1) - \log w'(e_2) + \cdots + \log w'(e_{k-1}) - \log w'(e_k)
\end{array}
\]

\( \Rightarrow \) alternating products in terms of \( w \) and \( w' \) agree.

\( \text{Rem: If } G \text{ is not planar, then } w \text{ and } w' \)

are gauge-equivalent \( \Rightarrow \)

\( \bullet \) the alternating product on all faces agree

\( \bullet \) the alternating product along any homology-cycle of the Riemann surface \( Y \) agree

\( \text{Moral: We later study transformations on weights that preserve the alternating products around faces/cycles.} \)
Kasteleyn weighting

Def: A Kasteleyn weighting of a planar bipartite graph is a choice of sign for each edge (thought of as an undirected edge) such that

Each face $F$ contains an odd # of $\pm$'s, if $F$ is a $4k$-gon

(*could also think of this as a Boltzmann weight if $F$ is a $(4k+2)$-gon*)

Aside:

One can also extend the notion of Kasteleyn weightings to be a choice of complex numbers $\mathbb{Z}$ with $|z|=1$ for each edge with the condition that

For each face $F$, the alternating product of weights

is in $\begin{cases} \mathbb{R} < 0 & \text{if } F \text{ is a } 4k \text{-gon} \\ \mathbb{R} > 0 & \text{if } F \text{ is a } (4k+2) \text{-gon} \end{cases}$

Since the alternating products of Kasteleyn weights around faces only depends on the size of faces, it follows that for planar graphs, all Kasteleyn weightings are gauge-equivalent.

Consequently, any two Kasteleyn weightings differ by a sequence of transformations given by multiplying all edges incident to a vertex by the same constant, i.e. (5).
Given a Kasteleyn weighting \( w \) on a planar graph \( G \), we define the Kasteleyn matrix \( K(G) \). Related to this weighting \( w \) is the \( (B|W) \) matrix

\[
K_{bw} = \begin{cases} 
0 & \text{if no edge } b \rightarrow w \\
1 & \text{if edge } b \rightarrow w \text{ in } E \\
\end{cases}
\]

(Here \( G = (V, E) \) with \( V = B \cup W = \{ \text{black vertices} \} \cup \{ \text{white vertices} \} \)).

**Example:**

\[
\begin{array}{cccc}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
\end{array}
\]

We now add extra weights to describe \( K \) more clearly.

\[
\begin{array}{cccc}
a & 1 & 0 & 0 \\
1 & b & 1 & 0 \\
1 & 0 & 1 & c \\
\end{array}
\]

**Claim:** Gauge transformations (i.e., equivalences) correspond to multiplying on the left and right by a diagonal matrix (depending on whether you are scaling at a black vertex or a white vertex).

**Thm:** \( Z = |\det K| \frac{1}{B = W} \).

**Example continued:**

\[
\det K = -a - c - abc
\]

\[
\Rightarrow Z = a + c + abc \Box
\]

For Boltzmann weights

\[
\begin{array}{cccc}
\alpha \downarrow \beta \downarrow \gamma \\
\alpha \downarrow b \downarrow c \\
1 & 1 & 1 & 1 \\
\end{array}
\]

\( \alpha \div b \div c \div 3 \) P.M.'s of \( G \)