Lecture 15: From polygons to bipartite tilings I: basic examples and triangles

Summary so far:

- Quiver + potential (satisfying AB condition)
- Unfold and dualize
- Bipartite tiling on torus
- K(\mathbb{Z}_2 \times \mathbb{Z}_2)
- Kasteleyn char poly and Newton polygon
- Toric diagram (convex with integer vertices)

Today: Goncharov-Kenyon construction
- Orbifolds/Hexagonal Lattice/twisted group ring

Sec 2.2 of Goncharov-Kenyon: Given convex polygon \( N \) in \( \mathbb{R}^2 \) with vertices in \( \mathbb{Z}^2 \), let \( \{ e_1, e_2, \ldots, e_k \} = \) primitive integer vectors \( (c_{aj}b) \) with \( \gcd(a_j, b) = 1 \) corresponding to sides of \( N \) in counter-clockwise order.

By construction, \( \sum e_c = 0 \).
A Torus $T$ can be represented as $\mathbb{R}^2/\mathbb{Z}^2$, so each $e_i^\mathbb{Z}$ determines a homology class $\bar{e}_i \in H_1(T, \mathbb{Z})$.

In fact, there is a unique geodesic (line in $\mathbb{R}^2/\mathbb{Z}^2$) representing this class of direction given by $e_i^\mathbb{Z}$.

Up to translations, place loops $(\alpha_{1j}, \ldots, \alpha_{kr})$ on $T$ so oriented direction of $\alpha_i$ is $e_i$ arranged so:

- No triple intersections (i.e., generic config.)
- Total number of intersection points is minimal.
- Alternating strand condition: Following loop $\alpha_i$ we encounter loops $\alpha_{ij1}, \ldots, \alpha_{ijd}$ in some order, we wish these to alternate crossing $\alpha_i$ from left-to-right to right-to-left or vice-versa.

$\alpha_{i1} \uparrow \alpha_{i2} \downarrow \alpha_{i3} \uparrow \alpha_{i4} \downarrow \ldots$  

Yields an admissible surface graph on torus $T$ whose $\mathbb{Z}$-cells are polygons oriented in one of 3 ways:

1) clockwise  
2) counterclockwise  
3) alternating
By replacing with white vertices and black vertices and contracting alternating regions, we get a new bipartite graph (using notions of Triple Point Diagrams going back to unpublished work of Dylan Thurston).

Examples:

\[
\begin{align*}
(-1,0) & \quad (0,1) \\
& \\
(0,0) & \quad (1,0) \\
& \\
& \\
(0,1) & \\
\end{align*}
\]

with \[e_1 = [-1, -1], \quad e_2 = [1, -1], \quad e_3 = [1, 1], \quad e_4 = [-1, 1]\]
Note that these clockwise and counter-clockwise regions are all quadrilaterals and correspond to 4-valent vertices.

SPP example:

\[
e_1 = [-1,-1] \\
e_2 = [1,0] \\
e_3 = [2,1] \\
e_4 = [-1,0] \\
e_5 = [-1,0]
\]

Note that face 2 is a hexagon.
We now focus on cases where \( \triangle \) is a triangle.

![Diagrams](image)

which shifts and straightens out to

Recover the hexagon e.g., of a bipartite tiling

Recall that corresponding \((q, w)\) was \(\mathbb{Z}^2\) with \(W = abc - acb\)

and superpotential algebra \(A = \mathbb{C}[a, b, c] / I_W\)

with \(I_W = \langle ab - ba, bc - cb, ac - ca \rangle\)

\(= A \cong \mathbb{C}[a, b, c]\)

3 perfect matchings which yield \(K(a, b, c) = \pm 1 \pm z_1 \pm z_2\) as well.

E.g., 1)

\(e_1 = e_2 = (0, -1), \quad e_3 = (1, 0), \quad e_4 = (-1, 2)\)
Point: bipartite tiling again involves only hexagons but fundamental domain is larger.

In particular, 2 faces rather than just 1

Perfect matchings by Kasteleyn matrix

$$\det K(z_1, z_2) = 1 + z_2 + z_2^2 - z_1 z_2^2$$

with Newton polygon

\[\begin{array}{c}
(0, 2) \\
(0, 1) \\
(0, 0)
\end{array}\]

which agrees w/ \(\bigtriangleup\) up to \(\text{GL}_2(\mathbb{Z})\)

Quiver and potential

\[W = ABD + CEF - ACB - DFE\]
Claim: \( \mathbb{A} \cong \text{twisted group ring } \mathbb{C}[x, y, z] \rtimes \mathbb{Z}_2 \).

Def: A twisted group ring \( S \times G \) (\( G \) acts on \( S \)) as an \( S \)-module defined with elements as pairs \( (s, g) \), \( s \in S, g \in G \) s.t.

\[
(s_1, g_1) \cdot (s_2, g_2) = (s_1, g_1 s_2) \cdot g_1 \cdot g_2
\]

where \( g_1 s_2 = \) result after \( g_1 \) left-acts on \( s_2 \).

In our e.g., let \( \mathbb{Z}_2 \) act on \( \mathbb{C}[x, y, z] \) by

\[
\begin{align*}
\varepsilon x &= -x \\
\varepsilon y &= -y \\
\varepsilon z &= z
\end{align*}
\]

\( \varepsilon \) represented by diagonal matrix \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \) in \( SL_3(\mathbb{C}) \).

We explain full method for equating \( \mathbb{C}[x, y, z] \rtimes G \) (\( G \) abelian \( \subseteq SL_3(\mathbb{C}) \))

when \( (\Omega, \mathbf{w}) \leftrightarrow \) triangular toric diagram later.

For now, draw unfolded quiver on hexagonal lattice.
Three cardinal directions

Define in CA

\[ X \rightarrow DF + CB \sim DF + BD \sim FC + BD \sim FC + CB \]

on 1  on 2

\[ Y \rightarrow DA + CE \sim \ldots \]

on 1  on 2

\[ Z \rightarrow AB + EF \sim \ldots \]

on 1  on 2

Equivalences in JW
Rem: multiplying by $X$ moves one unit in $x$-direction and moves face $\{1\}$ to $\{\geq\}$ and vice-versa.

multipl. by $Y$ "y-direction, and move face $\{1\}$ to $\{\geq\}$ and vice-versa.

BUT multiplying by $Z$ "z-direction, but face $\{1\}$, face $\{\geq\}$

corresponds to $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ diagonal action.

Exercise: 1) Show $xy = yx$, $xz = zx$, $yz = zy$ in $CQ/I\omega$.

2) Letting $P(x,y,z) \in \mathbb{C}[x,y,z] \subseteq \mathbb{C}[x,y,z] \times \mathbb{Z}_2$ come up with a way to define $(P(x,y,z), e) \mapsto (P(x,y,z), 1)$ so that multip in Twisted Group Ring agrees with that in $CQ/I\omega$.

3) Use this to complete argument $CQ/I\omega \cong \mathbb{C}[x,y,z] \times \mathbb{Z}_2$ for example of $Q$ and $W_0$.  