Lecture 19-20: F-terms, D-terms, Moduli space, and Master space

Recall from Lecture 12, we defined global symmetries as one parameter subgroups

\[ \mathcal{P} : \mathbb{C}^* \rightarrow \text{Aut}(A) \quad \left[ A = \text{superpotential algebra} \quad \mathcal{C} \left/ \mathcal{W} \right. \right] \]

s.t. for each \( t \in \mathbb{C}^* \), \( \mathcal{P}(t) \) defines the map

\[ \mathcal{P}(t) : \mathcal{X}_a \mapsto t^{v_a} \mathcal{X}_a \text{ for each } \mathcal{X}_a \in A \text{ corresponding to an arrow } a \in \mathcal{Q}_1. \]

\( \mathcal{P} \) is only well-defined if it acts homogeneously on terms in \( \mathcal{W} \).

Following Broomhead:

We let \( \mathcal{N}^+ \) be the cone of \( \mathcal{V} \)'s with all entries nonnegative and proved that perfect matchings of the bipartite tiling corresponding to \( (\mathcal{C}, \mathcal{W}) \) were the generators of \( \mathcal{N}^+ \) by

\[ \text{perfect matching } \mathcal{M} \mapsto \mathcal{V}_\mathcal{M} = \text{0-1 vector with value 1 on } a \in \mathcal{Q}_1 \text{ corresponding to } e_a \text{ in } \mathcal{M}. \]

Today, we explain how \( \mathcal{N}^+ \) is related to the toric diagram \( \Delta \) corresponding to

- Kasteleyn char poly of \( \mathcal{N}^+ \)'s generators

Using choice of torus domain for our bipartite tiling,
Let $M = \{Q_i\} \times m$ $(0-1)$-matrix with rows indexed by edges $(e_1, \ldots, e_k)$ of the bipartite tiling and columns indexed by perfect matchings $(P_1, \ldots, P_m)$.

We let entry $M_{ij} = 1$ if perfect matching $P_j$ contains $e_i$; 0 otherwise.

Matrix $M$ will almost always have linear dependencies in the columns which we record in the $F \times m$ matrix $Q_F$ defined as $Q_F = (\ker M)^T$.

Additionally, for any face $F_k$ of our bipartite tiling, we define $P_k^+$ and $P_k^-$ as the perfect matchings which each use exactly half of the edges around face $F_k$ and look the same outside of this face. (We in particular let $P_k^+$ look like $F_k^+$ and $P_k^-$ look like $F_k^-$.)

And extend arbitrarily to a perfect matching on the rest of the torus.

Remark: There are examples where constructing such $P_k^+$ and $P_k^-$ are not possible. However, the following construction still works after defining variants of $P_k^+$ and $P_k^-$. 

4/6/15 (c)
$P_i^+$ should contain edges $A_jD_j$ and $F_i$.

However, if in this e.g. the full domain only contains two black vertices & two white vertices,

1) we would need to include both copies of edge $D$ incident to face 1, which result in two edges ($0D$ & $F_i$) incident to the bottom vertex.

2) We will come back to examples like this.

Consider the linear combination $P_k^+ - P_k^-$ for every face $F_k (k=1,...,1001)$ which results in:

- $F_k$ and all edges are cancelled outside of $F_k$.

In degenerate e.g.'s like above, we abuse notation and let $P_i^+ = A + D + F_i$ and $P_i^- = B + D + E$.

Then $P_i^+ - P_i^- = A + F_i - B - E$ (the two $D$'s cancel).

Which separates the strip of faces 1 from the strip of faces 2.

Furthermore, agrees with alternating sum $A - B + D - E + F - D$ around $F_i$. 

$4/6/15$
Ignoring this degeneracy issue for the moment, we get \( \mathbb{Z} \)-linear combinations of perfect matchings corresponding to alternating sum around each face \( F_k \). We obtain a flow \( \bullet \to \circ \) for pos, \( \bullet \leftarrow \circ \) for neg.

\( (1 \mathbb{Q}_0 - 1) \) of these are linearly independent since we can always define the outside of \( F_k \) to be the contour around every other face.

E.g. 4-checkerboard

Let \( \mathbb{Q}_0 = (1 \mathbb{Q}_0 - 1) \times m \) matrix by picking all but one face and writing alternating sum around it as \( \mathbb{Z} \)-linear combo of perfect matchings.

Rem.: As we will see later, even if \( P_{k+} - P_k \) construction does not work, can always isolate a single face (alternating sum of its edges) as a \( \mathbb{Z} \)-linear combin. of \( P_i \)'s.
4/6/15 5. Build matrix $Q = \begin{bmatrix} Q_D \\ Q_E \end{bmatrix}$

$m = \# $ perfect matchings

Claim: For a bipartite tiling on a torus

$(\text{ker } Q)^T = 3 \times m$ matrix

such that in row-echelon form,

each column sums to 1.

⇒ cols of $(\text{ker } Q)^T$ are coplanar.

Projecting to the plane

Claim: columns are vertices of tonic diagram $\Delta$

with multiplicities.

Physics literature discusses a proof in

"Moduli Spaces of Gauge Theories from Dimer Models: Proof of the Correspondence" by Franco and Veych (arXiv:0601063)

After some examples, I will follow the proof in

[Broomhead, sec 2.3] using different terminology.
5 perfect matchings: \( AE, AF, BE, BF, CD \)

\[
\begin{array}{cccccc}
\text{P} & \text{P}_2 & \text{P}_3 & \text{P}_4 & \text{P}_5 \\
\text{A} & 1 & 1 & 0 & 0 & 0 \\
\text{B} & 0 & 0 & 1 & 1 & 0 \\
\text{C} & 0 & 0 & 0 & 0 & 1 \\
\text{D} & 0 & 0 & 0 & 0 & 1 \\
\text{E} & 1 & 0 & 1 & 0 & 0 \\
\text{P} & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

\[
Q_F = (\ker M)^T = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 \end{bmatrix}
\]

As discussed earlier, \( \text{Face } 1 \mapsto \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \end{bmatrix} \)

AF-BE

in fact, \( \text{Face } 2 \mapsto \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \end{bmatrix} \)

(And clear linear dependence)

\[
Q = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 \\
0 & -1 & -1 & 1 & 0 \\
\end{bmatrix}
\]

\[
(\ker Q)^T = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[\mapsto \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]


\[\sim\]

\[\text{up to translation and } GL_2(\mathbb{Z})\]
4/6/15 Example 2 \( C^3/\mathbb{Z}_3 \)

six perfect matchings: \( M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \)

\( Q_p = \text{ker } M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \)

\( F_0 = g - c + i - b + e - a = p_6 - p_4 \)

\( F_1 = d - e + f - g + h - c = p_5 - p_6 \)

\( F_2 = b - h + a - d + c - f = p_4 - p_5 \)

\( Q_p = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad Q = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \)

\( \text{with } (\text{ker } w)^T = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{bmatrix} \quad \text{row-reduce} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \mathbb{I} \)

Rewrite in words (1)-(2) \[ [1-10000] \]
(2)-(3) \[ [01-1000] \]
Example 3

Graph:

\[\begin{align*}
&\text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} & \text{G} & \text{E}_A \\
\text{A} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{B} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{C} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\text{D} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\text{E} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\text{F} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\text{G} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{align*}\]

This graph has a perfect matching matrix:

\[M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}\]

(see pg. 12 of Lecture 12)

\[Q_F = (\ker M)^T = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}\]

\[Q_0 = \begin{bmatrix} -1 & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 2 & 1 & 1 & 0 \end{bmatrix}\]

\[Q = \begin{bmatrix} 0 & 0 & 1 & -1 & -1 & 1 \\
-1 & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 2 & 1 & 1 & 0 \end{bmatrix}\]

\[\ker Q = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}\]

\[c_1 + c_2 \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 2 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}\]

Rem: This is an example where \( p_+ = p_- \) construction not sufficient.
We now turn to \cite{Broomhead} for proof of the following result:

**Claim:** For a bipartite tiling on the torus, building the matching matrix $M_j$ and associated matrix $Q$ (built from $Q_F = \ker (M)^T$ and $Q_0$), then $(\ker Q)^T$ is of rowdim 3 and its columns are coplanar on $xt+yt+z=1$.

Lastly, projecting to the plane yields toric diagram $\Delta$, agreeing with the Newton polygon (including multiple of Kasteleyn characteristic polynomial $h(3w)$).

Before giving the proof, we rephrase this algorithm in terms of Broomhead's language of algebraic topology and commutative algebra.

1) cone $N^+ = \{Q\}$ is generated by $(0,1)$-functions corresponding to perfect matchings.

2) By construction, matching matrix $M$ has columns corresponding to generators of $N^+$, written in $\mathbb{Z}^2$-vector form.

However, the generators (as given) satisfy $\mathbb{Z}$-linear relations, i.e., they do not freely generate $N^+$. 
4/6/15 10  \[ Q_F = (\ker M)^T \] encodes these relations.

For \( C/\mathbb{Z}_2 \times C \)

\[ N^+ \text{ gen'd by } p_1 = AE, p_2 = AF, p_3 = BE, p_4 = BF, p_5 = CD \]

but we have the relation \( p_1 p_4 = A E B F = p_2 p_3 \)

if we think of these as funs on \( \mathbb{Z}_2^\times \)

so \( A E B F = A F B E \) is shorthand for

function \( g \) s.t. \( g(A) = 1, g(B) = 1, g(C) = 0, g(D) = 0, g(E) = 1, g(F) = 1 \).

Thus \( (\ker Q_F)^T \) yields generators which freely generate \( N^+ \).

In the e.g. \( (\ker Q_F)^T = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

so \( N^+ \) freely generated by \( \{ v_1, v_2, v_3, v_4 \} \) where

\[ p_1 = v_1 = AE, p_2 = v_2 = AF, p_3 = v_3 = BE, \]

\[ p_4 = \frac{\sqrt{3}}{2} \frac{v_3}{v_1} = \frac{(AF)(BE)}{(AE)} = BF, p_5 = v_4 = CD \]

\[ \Rightarrow \quad v_1 = AE, v_2 = AF, v_3 = BE, v_4 = CD \]
4/6/15 11: 3) We build an exact sequence

\[ 0 \to \mathbb{Z} \to \mathbb{Z}^{\mathbb{Q}_0} \xrightarrow{d} N \to N_0 \to 0 \]

where map \( d \) is the cochain map defined by

\[ dF \in \mathbb{Z}^{\mathbb{Q}_0} \] defined as \( dF(a) = f(h) - f(t) \)

an arrow \( \in \mathbb{Q}_1 \), \( t \to h \), \( t \in \mathbb{Q}_0 \)

In particular, to insure \( \text{im} (\mathbb{Z} \to \mathbb{Z}^{\mathbb{Q}_0}) = \ker d \)
we define \( \mathbb{Z} \to \mathbb{Z}^{\mathbb{Q}_0} \) s.t. \( f(v) = \lambda \) for every \( v \in \mathbb{Q}_0 \).

Clearly, \( dF = 0 = (\lambda - \lambda) \) on every arrow in this case.

\[ \text{im } d \in N \text{ since } \mathbb{Z}^{\mathbb{Q}_0} \xrightarrow{d} \mathbb{Z}^{\mathbb{Q}_1} \xrightarrow{d} \mathbb{Z}^{\mathbb{Q}_2} \]

is a cochain complex satisfying \( d^2 = 0 \)

have any \( g \in \text{im} (\mathbb{Z}^{\mathbb{Q}_0} \xrightarrow{d} N) \) also in \( d^1(0) \)

and \( N \) was defined as \( d^1(\mathbb{Z}) \).
Thus all maps in this exact sequence are well-defined, including $N \rightarrow N_0$ which is the surjection onto $(\text{coker } d) = \overline{\text{im } d}$.

Remark: Thinking of these as automorphisms of the path algebra (if we think of the related torus actions, i.e., one parameter subgroups) Broomhead refers to $\mathbb{Q}_0$ as $N_{\text{in}}$ (inner automorphisms) and $N/\text{im } d$ as $N_0$ (outer automorphisms).

As Broomhead notes, physics literature also calls $N_{\text{in}}$ as baryonic symmetries and $N_{\text{out}}$ as mesonic symmetries.

(See Kenna, Section 3.6.1)

4) As discussed in (2), using $(\ker(Q_\mu))^\perp$, we have a set of generators $(v_1, \ldots, v_K)$ that freely generates cone $N^+$. Thus $N_0^+ := N^+ \cap \left( \frac{N_0 \times \mathbb{R}}{\mathbb{Z}} \right)$ obtained by imposing new relations coming from $\text{im } d$.

(See p.16, saturation of the projection of the cone $N^+ \cap \overline{N}$ into the $(2g+1)$-rank lattice $N_0$ where $g = \text{genus of torus}$.)
E.g., continued, starting with \( F \in \mathbb{Z} \) defined by

\[
F(1) = \lambda_1, \quad F(2) = \lambda_2,
\]

then \( dF \in \mathbb{R}^{\mathbb{N}} \) defined by

\[
F(A) = \lambda_2 - \lambda_1, \\
F(B) = \lambda_1 - \lambda_2 = \text{need } dF = g \in \mathbb{R} \text{ s.t.}, \\
F(C) = \lambda_2 - \lambda_2 = 0, \\
F(D) = \lambda_1 - \lambda_1 = 0, \\
F(E) = \lambda_1 - \lambda_2, \\
F(F) = \lambda_2 - \lambda_1
\]

In terms of our freely generating set

\[
v_1 = AE, \quad v_2 = AF, \quad v_3 = BE, \quad v_4 = CE
\]

\[
\Rightarrow \quad g(v_1) = 0, \quad g(v_2) = 2(\lambda_2 - \lambda_1), \\
g(v_3) = 2(\lambda_1 - \lambda_3), \quad g(v_4) = 0
\]

i.e., we quotient by functions \( g \) multiples of the form

\[
\begin{bmatrix}
0 & 1 & -1 & 1
\end{bmatrix}
\]

Notice this exactly matches \( \mathbb{Q}^+ \) for this e.g.

(We will explain why momentarily)

\[
\begin{bmatrix}
g(p_1) & g(p_2) & g(p_3) & g(p_4) & g(p_5)
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
0 & 1 & -1 & 0 & 0
\end{bmatrix}
\]

Using larger (non-free) generating set for \( \mathbb{N}^+ \).
In conclusion, taking \((\ker Q_0)_{\text{gens}}\) yields rows which give generators for cone \(N_0^+\) in terms of free gens \((v_1, \ldots, v_6)\):

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & g_1 \\
0 & 1 & 0 & 0 & 0 & g_2 \\
0 & 0 & 0 & 0 & 1 & g_3
\end{bmatrix}
\]

\(g_1 = g_1(v_1) = 1, g_1(v_2) = 0, g_1(v_3) = 0, g_1(v_4) = 0, g_1(v_5) = 0, g_1(v_6) = 0\)

\(g_2 = g_2'(v_1) = 0, g_2'(v_2) = 1, g_2'(v_3) = 1, g_2(v_4) = 0, g_2(v_5) = 0, g_2(v_6) = 0\)

\(g_3 = g_3'(v_1) = 0, g_3'(v_2) = 0, g_3'(v_3) = 0, g_3'(v_4) = 0, g_3'(v_5) = 0, g_3'(v_6) = 0\)

\(g_1, g_2, g_3\) generate \(N_0^+ = \mathbb{R}^Q\)

We now rewrite \(g_1, g_2, g_3\) in terms of \(p_1, p_2, \ldots, p_6\):

\(g_1(p_1) = 1, g_1(p_2) = 0, g_1(p_3) = 0, g_1(p_4) = g_1(v_2) + g_1(v_3) \quad g_1(p_5) = g_1(v_4) = 0\)

\(g_2(p_1) = 0, g_2(p_2) = 1, g_2(p_3) = 1, g_2(p_4) = 2, g_2(p_5) = 0, g_2(p_6) = 0\)

\(g_3(p_1) = 0, g_3(p_2) = 0, g_3(p_3) = 0, g_3(p_4) = 0, g_3(p_5) = 0, g_3(p_6) = 0\)

Similarly, \(g_2(p_1) = 0, g_2(p_2) = 1, g_2(p_3) = 1, g_2(p_4) = 2, g_2(p_5) = 0\)

We could also obtain \(g_1, g_2, g_3\) in terms of the \(p_i\)'s directly by taking \((\ker Q)^T\) where

\(Q = \begin{bmatrix} Q_E & \vdots & Q_D \end{bmatrix}\) with columns given by the \(p_1, \ldots, p_m\)’s.
Left to show

- Show the set \( \{ \mathbf{m} \mid \mathbf{m} \in \mathbb{Z}^d \} \subseteq \mathbb{Z}^d \) is the \( \mathbb{Z} \)-linear combination of \( F_i \), where
  \[
  F_i(e) = \begin{cases} 
  1 & \text{if } e = i \\
  -1 & \text{if } e = -i \\
  0 & \text{otherwise}
  \end{cases}
  \]

- Show \( \ker \mathbb{Q} \) (or equivalently \( \ker \mathbb{Q} d \mathbb{Z} \)) has 3 generators whose sum is a constant function.

  (Equivalently, want to show that \( N_0 \mathbb{Q} \) is a cone whose 3 generators are coplanar.)

- Terms of \( K(\mathbb{Z}) \) indeed correspond to the coordinates of these generators.

Phrased in this way the first part is clear since each \( F_i \in \mathbb{Q}^d \) defined by \( F_i = d \mathbf{e}_i \), where

- \( \mathbf{e}_i \in \mathbb{Q}^d \) satisfies \( g_i(j) = 1 \) for \( j = i \)
- \( g_i(j) = 0 \) for \( j \neq i \).

Thus \( \mathbb{Z} \)-linear combus exactly \( \mathbf{m} \in \mathbb{Z}^d \).
In fact this shows that we can always write an alternating sum of edges around a face as a \( \mathbb{Z} \)-linear combo of perfect matchings (even if \( P^+_K - P^-_K \) construction fails).

Since such a function in \( (\text{im } d) \subset N^+ \) and the cone \( N^+ \) is generated by the perfect matchings.

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For the second part, Broomhead constructs the short exact sequence

\[
0 \to H^1(Y, \mathbb{Z}) \to N_0 \xrightarrow{\text{deg}} \mathbb{Z} \to 0
\]

where in our case, \( Y = \text{torus} \) rather than an arbitrary Riemann surface.

\[
\Rightarrow H^1(Y, \mathbb{Z}) \cong \mathbb{Z} \;
\]

in our case.

Since perfect matchings are of degree 1 in \( N_0 \) and generate \( N_0^+ \)

\[
\ker \left( \xrightarrow{\text{deg}} N_0 \to \mathbb{Z} \right) \text{ generated by } \{ P_2 - P_1, P_3 - P_1, ..., P_m - P_1 \} \text{ where } p_i \text{ is chosen arbitrarily.} \]
4/6/15 17 e.g. cont. (see pg. 6)

\[ p_2 - p_1 = F - E \]

\[ p_3 - p_1 = B - A \]

\[ p_4 - p_1 = B + F - A - E \]

\[ p_5 - p_1 = C + D - A - E \]

Called height functions

\( \text{Noted in each case} \)

\[ (h_x, h_y) \]

\[ (h_x, h_y) \]

Also \( p_1 - p_1 = (0, 0) \)

\[ \text{Negative contribution for crossing } (E_k) \text{ (red)} \]

\[ \text{Positive contribution for crossing } (E_k) \text{ (blue)} \]
be a linear combination of fundamental cycles (rather than as contours around faces).

we conclude that \( N_0 \) is a rank 3 lattice (rank \( 2y+1 \) for general Riemann surf.)

perfect matchings all have degree 1 so their images in \( N_0 \) span a lattice polytope in a rank 2 affine sublattice, \( N_0^* \) is the core on this polytope.

\((p_1 - p_2, p_2 - p_3, \ldots, p_m - p_1)\) are each cocycles and the lattice polytope is the convex hull of all the relative cohomology classes (of the cocycles) with multiplicities.

Next time: Zig-Zags and a different set of cocycles.

Master space = \( (\text{Ker } Q_F)^\perp \leftrightarrow \text{ matching polytope} \)

Moduli space = \( (\text{Ker } [Q_F \mid Q_0])^\perp \leftrightarrow \text{ matroid polytope} \)

See = Matching polytopes, toric geometry, and the nonnegative Grassmannian" by Pustnkhov, Speyer, Williams
4/6/15 18. We think of $H^1(Y; \mathbb{Z})$ as $\ker d: \mathbb{Z}^q_1 \rightarrow \mathbb{Z}^q_2$ \[ \frac{\text{Im } d: \mathbb{Z}^q_0 \rightarrow \mathbb{Z}^q_1}{\text{Im } d: \mathbb{Z}^q_0 \rightarrow \mathbb{Z}^q_1} \]

which on the level on functions on edges of a bipartite tiling (rather than on arrows of the quiver) is the quotient

\[ \frac{\{ \text{Functions that sum to zero at every vertex} \}}{\{ \text{Functions that alternate around faces of the tiling} \}} \]

E.g., and zero every other edge is zero at every vertex, but is in $\text{Im } d: \mathbb{Z}^q_0 \rightarrow \mathbb{Z}^q_1$.

\[ N_0 = N / \text{im } d: \mathbb{Z}^q_0 \rightarrow \mathbb{Z}^q_1 = \mathbb{Z}^q_1 / \text{im } d \]

so $\ker : N_0 \rightarrow \mathbb{Z} = \ker: \mathbb{Z}^q_1 \rightarrow \mathbb{Z}^q_2$.

Indeed is $H^1(Y; \mathbb{Z})$.

In the case of the torus, $H^1(Y; \mathbb{Z}) \cong \mathbb{Z}^2$ are the two directions of fundamental cycles.

In other words, we want $\mathbb{Z}$-linear combos of perfect matchings to