Lecture 27: Proofs of Speyer's Octahedron Rec. Combinatorial Interpretation

Proof 1: Kuo Condensation

We begin by summarizing Eric Kuo's 2003 paper "Applications of Graphical Condensation for Enumerating Matchings and Tilings" (arXiv:0304090)

**Thm 2.1** Let $G = (V, E)$ be a planar bipartite graph with $|V| = |W|$, with a specific planar embedding in mind so there is a cyclic ordering of edges around each vertex.

Let vertices $a, b, c, d$ appear in a cyclic order on a face around $G$ (possibly the infinite face).

Then

\[ M(G)M(G-(a,b,c,d)) = M(G-(d))M(G-(a,c,d)) + M(G-(c,d))M(G-(b,c)) \]

where $M(G) = \#$ perfect matchings in $G$.

Also $G' = G-(a,b)$ means delete vertices $a$ & $b$ from $G$ along with any incident edges.
To prove this theorem, Kuo considers the superposition of a matching of $G$ with a matching of $G - \{a, b, c, d\}$ versus a superposition of matchings of $G - \{a, b, d\}$ and $G - \{c\}$.

OR a superposition of matchings of $G - \{a, d\}$ and $G - \{b, c\}$.

Each of these superpositions is a collection of edges (possibly with duplicates) so that vertices $a, b, c, d$ are incident to exactly one edge but every other vertex (in $G - \{a, b, c, d\}$) is incident to exactly two edges in the collection.

Schematically, such superpositions look like:

The paths either go $a \rightarrow b \rightarrow c$ (as pictured) or $d \rightarrow c$.
Note that these two paths are both of odd length so splits into two perfect matchings

\[ \begin{array}{c}
\text{can be split into 2 different perfect matchings}
\end{array} \]

where one is incident to \( s_a, b \) and one is not.

We thus obtain a \( 2^k \) to \( 2^k \) map (where \( k = \# \text{cycles} \)) with \( \geq 4 \) vertices

\[ M(G)M(G-s_{a,b,c,d}) \rightarrow M(G-s_{b,c})M(G-s_{a,d}) \]

\[ \cup M(G-s_{a,d})M(G-s_{b,c}) \]

Rem: Cannot connect \( a \) to \( b \) because then the paths would have to intersect in a vertex of degree \( > 2 \).

Variant (Thm 2.3) Assume \( |B| = |W| \) & Ensure the colors are different then above

\[ M(G)M(G-s_{a,b,c,d}) = M(G-s_{a,d})M(G-s_{b,c}) - M(G-s_{a,c})M(G-s_{b,d}) \]

PF Sketch: Superposition involves either

- even length paths

\[ \begin{array}{c}
\text{even length paths}
\end{array} \]

\[ \begin{array}{c}
\text{odd length paths}
\end{array} \]

OR

- odd length paths
Again, paths do not exist since no vertices of degree \( \geq 2 \).

Even length paths:

\[
\text{decompose as matchings as one in } G - \{a, d\} =
\& \text{ one in } G - \{b, c\}
\]

\text{OR}

\[
\text{one in } G - \{a, c\} =
\& \text{ one in } G - \{b, d\}
\]

Odd length paths:

\[
\text{decompose as matchings as one in } G - \{a, d\}
\& \text{ one in } G - \{b, c\}
\]

\text{OR}

\[
\text{one in } G
\& \text{ one in } G - \{a, b, c, d\}
\]

Besides the cycles, we again have a bijection but rearranged:

\[
M(G - \{a, d\}) M(G - \{b, c\}) \rightarrow M(G - \{a, c\}) M(G - \{b, d\})
\]

\[
\cup M(G - \{a, b, c, d\}) M(G)
\]
Lastly, there are unbalanced versions of Kuo condensation.

**Thm 2.4** \[ |B| = |W| + 1 \]

\[
M(G-S_b)M(G-S_{a,c}d) = M(G-S_{a,b})M(G-S_{b,c}d) + M(G-S_c)M(G-S_{a,b}d) \]

**PF Sketch:**

\[
M(G-S_b)M(G-S_{a,c}d) \text{ or } M(G-S_c)M(G-S_{a,b}d) \]

\[
M(G-S_{a,c})M(G-S_{a,b}d) = M(G-S_{a,b})M(G-S_{c,d}) + M(G-S_{a,d})M(G-S_{b,c}) \]

\[
M(G-S_{a,c})M(G-S_{a,b}d) \text{ or } M(G-S_{a,c})M(G-S_{a,b}d) \]

**Thm 2.5** \[ |B| = |W| + 2 \]

\[
M(G-S_b)M(G-S_{a,c}d) \text{ or } M(G-S_c)M(G-S_{a,b}d) \]
Kuo also discusses weighted Aztec Diamonds (Thm 5.5)

\[ m(A_n) m(A_{n-2}) = m(A_{n-1})^2 + m(A_{n-1})^2 \]

up to including some extra edges (with weights)

\[ w(A_n) w(A_{\text{middle}}) = \text{lor} w(A_{\text{top}}) w(A_{\text{bot}}) \]

\[ + \text{to b } w(A_{\text{left}}) w(A_{\text{right}}) \]

Where

\[ \text{PF is analogous to above. We will discuss a} \]

\[ \text{more general case later and its proof then.} \]
Specner merges some of these cases into one
Condensation Theorem

Thm: Let $G$ be a bipartite planar graph with vertices
partitioned into nine sets (disjointly);

$V(G) = \{C, LIN, M, E, USL, USW, UWN, NW\}$

satisfying the following conditions:

1) Edges bridging between two of these regions must be
colored as follows:

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In other words, edges incident to NW or SE (and another region)
must be black,
edges "" NE or SW must be white
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Note also that two regions can be connected by an edge
only if the pair is an ordinary direction and adjacent cardinal direction
where we also consider C to be a "cardinal direction"
The regions NW & SE contain one more black vertex than white vertex

The regions NE & SW have more white vertex than black vertex.

The other five regions are balanced.

Then

\[
M(G)M(C) = M(WUNWUSWUC)M(EUNEUSEUC)M(S)M(N)
\]

\[
+ M(SUSWUSEUC)M(NUNWUNNEUC)M(W)M(E)
\]

Remark: Here are some examples how Speyer's version specializes to Knutson's:

Example 1) is a 4-face. Let

\[
\begin{align*}
NW &= \{a\}, \quad NE = \{b\} \\
SW &= \{d\}, \quad SE = \{c\} \\
C &= G - \{a, b, c, d\} \\
N &= S = E = W = \emptyset
\end{align*}
\]

Then

\[
M(G)M(G - Sa,b,c,d) = M(G - Sc,d)M(G - Sa,d) + M(G - Sa,b,c)M(G - Sc,d)
\]

Variant: \(Sa,b,c,d\) on >4-face, e.g.,

assign regions just as before but now

NW/C = E/NE

SW/C = SE

with no connections between ordinal directions.

Combination of these = Thm 2.1 in general.
Weighted Aztec Diamond case could be treated as ordinary Kuo condensation (Thm 2.1) or via Speyer's reformulation.

E.g.,

\[ G - (a, b, c, d) \]

\[ G - (a, d) \quad (G - (b, c) \text{ analogous}) \]

\[ G - (a, b) \quad (G - (c, d) \text{ analogous}) \]

\[ \text{VERSUS} \]

\[ m(G) m(C) = m(NUNWUNEUC) \]

\[ m(SUSWUSEUC) \]

\[ m(E) m(W) \]

\[ + \quad m(EUNEUSEUC) \]

\[ m(WUNWUSEUC) \]

\[ m(N) m(S) \]
We will prove Speyer's formulation of condensation shortly.

First, we describe how it proves his Octahedron Recurrence combinatorial interpretation.

Consider $G(n, j, i, j)$'s as constructed last lecture.

By abuse of notation, let $V(n, j, i, j)$ denote the vertices of $G(n, j, i, j)$.

To avoid a boundary case, assume $n$ large enough so that $(n - 2, i, j) \notin C$.

Claim 1: $V(n - 2, i, j) = V(n - 1, i + 1, j) \cap V(n - 1, i - 1, j)$

Claim 2: $V(n, j, i, j) = V(n - 1, i + 1, j) \cup V(n - 1, i - 1, j) \cup V(n - 1, i, j + 1) \cup V(n - 1, i, j - 1)$

Let $K = G(n - 2, i, j)$.

Let $NE = G(n - 1, i + 1, j) \cap G(n - 1, i, j + 1) - C$

Let $NW = G(n - 1, i - 1, j) \cap G(n - 1, i, j + 1) - C$

Let $N = G(n - 1, i, j + 1) - (NE \cup NW \cup C)$

Define the remaining regions analogously.
Claim 3: Region \( N \) (and analogously regions \( E, W, J, S \)) has a unique perfect matching.

Claim 4: Regions \( NE \) and \( SW \) have one more black vertex than white. Regions \( NW \) and \( SE \) have one more white vertex.

Claim 5: The adjacencies between the nine regions is as needed for the hypotheses.

Result: \( F(n; i, j) F(n-2; i, j) = \)

\[ a \cdot c \cdot f(n-1; i, j; i+1) \cdot f(n-1; i, j; i-1) \]
\[ + b \cdot d \cdot f(n-1; i; j+1) \cdot f(n-1; i; j-1) \]

where \( a, b, c, d \) are weights of edges in the unique perfect matchings of region \( E, N, W, J, S \), resp.

Technically, Speyer proves this by letting face weights go to 1 and introducing a different edge weighting, but a, c, b, d exactly compensate for cancelling out face weights for faces in \( G \), but not a particular region, \( G(J, i, j) \).
Example \((\text{Sumos - 4})\)

\[ G_{0, -\frac{3}{2}} = \]
\[ -3 \]
\[ -3 \]
\[ -4 \]
\[ -4 \]
\[ -4 \]
\[ -4 \]
\[ -5 \]
\[ -5 \]
\[ -5 \]
\[ -5 \]
\[ -6 \]
\[ -3 \]
\[ -4 \]
\[ -3 \]
\[ -3 \]
\[ -2 \]
\[ -2 \]

while we consider \( G_{-\frac{3}{2}, -\frac{3}{2}}, G_{-\frac{1}{2}, -\frac{5}{2}}, G_{-1, -\frac{5}{2}}, G_{-1, -\frac{1}{2}}, G_{1, -\frac{6}{2}} \)

\[ G_{-\frac{3}{2}, -\frac{3}{2}} = \]
\[ -3 \]
\[ -3 \]
\[ -3 \]
\[ -3 \]

\[ G_{-\frac{5}{2}, -\frac{3}{2}} = \]
\[ -3 \]
\[ -3 \]
\[ -3 \]
\[ -3 \]

\[ \Rightarrow C \]
Verifying Claims

Claim 1: $C(n-2, c, j, i)$ includes faces taxicab distance away from $(n-2, c, j, i)$ less than the difference $(n-2) - n'$

$G(n-1, i, j, i)$ or $G(n-1, j, i, i)$ similar but as if one step down further is possible.

But will go E/W or N/S hence why intersections match up.

Claim 2: Similarly $G(n, c, j, i)$ compared w/

$G(n-1, i, j, i)$ or $G(n-1, j, i, i)$ allows one more step down so their union yields everything.

Claim 3: On border so is at best a "thickened path" possible.

Claim 4 & 5: By Claim 3, this unique perfect matching is a path which we can augment on both sides using one vertex of opposite colors using vertices in the adjacent ordinal regions.

Pairing all these off & the fact that $G, C, E, W, S, N$ are colored balanced yields the right adjacencies counts.

Augmentation: $x \rightarrow y \rightarrow x \rightarrow y$
Sketch of PF of Speyer's Condensation Thm

Similar to Kuo: Under the hypotheses

Superimposing a matching on $G$ and $C$ leads to $\chi_1 \cup \chi_2 \cup M_c \cup M_q \subseteq X$

where $\chi_1$ is a $N$-join, $\chi_2$ is a $S$-join

OR $\chi_1$ is a $E$-join, $\chi_2$ is a $W$-join

$M_c = \text{disjoint union of cycles in } C$

$X = \text{eight cardinal and ordinal regions}$

and $M_q = \text{matching entirely inside } \mathbb{C}_{\text{NE,E, etc.}}$

A $N$-join is (Others analogous)

1) a path $\text{NE} \xrightarrow{} \text{NW}$ through $C$

2) a single edge $\text{NE}$ to $\text{NW}$

OR 3) a pair of edges

$LHS = M(G)M(c), \quad RHS = \quad + \quad N$-join + $S$-join E-join + $W$-join