Lecture 28: Proofs of Speyer's Octahedron Recurrence

Combinatorial Interpretation II

Theorem: Build $G(n, i, j)$'s from infinite graph $G$ as discussed last week. For $(n_0, i_0, j_0)$ in $G$, let $F(n_0, i_0, j_0) = x_{i_0 j_0}$.

Assume the other $F(n, i, j)$'s satisfy the octo rec.

$$F(n, i, j) F(n-2, i, j) = F(n-1, i+1, j) F(n-1, i-1, j) + F(n-1, i, j+1) F(n-1, i, j-1).$$

Then for $n > h(i, j)$, we have

$$F(n, i, j) = \sum w(M) \text{ using the face weight}$$

$$M \text{ a perfect matching of } \text{discussed last week}$$

$$G(n, i, j)$$

Fix $(n_0, i_0, j_0)$ such that $n_1 \geq h(i_0, j_0)$.

Proof 2 (by Urban Renewal):

Let $U = \{ (n, i, j) : n + i \leq 0 \mod 2 \}$ $\cap h(i, j) < n$}

$$C(n, i, j) = \{ (n, i, j) : n + i \leq 0 \mod 2 \}$ $\cup n \leq n_1 - |i - i_0| - |j - j_0|$$

(Closed cone)
We prove the Theorem by induction on the number of faces in \( \bigcup \mathcal{C}_{(n_1, i_1, j_1)} \).

If \( \bigcup \mathcal{C}_{(n_1, i_1, j_1)} = \emptyset \) (and we assume \( n_1 \geq h(c_{i,j}) \))

then \( n_1 = h(c_{i,j}) \), i.e. \( (n_1, i_1, j_1) \in \mathcal{C} \) and

\[ f(n_1, i_1, j_1) = x_{i_1, j_1} \text{ by hypothesis.} \]

Otherwise, define \( \tilde{h} : \mathbb{Z} \rightarrow \mathbb{Z} \) by

\[
\tilde{h}(c_{i,j}) = \min \left( h(c_{i,j}), n_1, -\lfloor i - c_{i,j} \rfloor, -\lfloor j - c_{i,j} \rfloor, 1 \right)
\]

E.g., \( h(c_{i,j}) = \begin{cases} 0 & \text{if } c_{i,j} \equiv 0 \mod 2 \\ -1 & \text{if } c_{i,j} \equiv 1 \mod 2 \end{cases} \)

then \( \tilde{h}(c_{i,j}) \) relative to \( (n_1, i_1, j_1) = (0, 0, 0) \)

\[
\begin{array}{ccc}
-4 & -3 & -4 \\
-3 & -1 & 0 & 2 & -3 & 1 & 2 & -4 \\
-2 & -1 & 0 & 1 & -2 & 1 & 1 & -3 & 1 & -4 \\
-1 & -1 & 0 & 0 & 0 & -1 & 0 & -2 & 0 & -3 & 0 & -4 \\
-2 & -1 & 0 & -1 & 0 & -1 & 0 & -2 & -1 & 0 & -2 & -3 & -4
\end{array}
\]

i.e.
On the other hand, \( \widetilde{h}(i, j) \) (relative to \( (n_{ij}, i, j) = (-2, j, 1) \))
looks like

\[
\begin{array}{cccc}
-6 & -5 & -4 & -5 \\
-5 & -4 & -3 & 1, 2 & -4 \\
-4 & -3, 0, 1 & -2, 1 & 1 & -3, 2, 1 \\
-5, 0, 1 & -4, 0, 0 & -3 & 1, 0 & -4 \\
-6 & -5, 0, 1 & -4 & 1 & -1 & -5 \\
\end{array}
\]

while \( \widetilde{h}(i, j) \) (relative to \( (n_{ij}, i, j) = (+2, j, 1) \))
looks like

\[
\begin{array}{cccccccc}
& -4 & -3 & -2 & -1 & -2 & -3 & -4 \\
-3 & -2 & -1 & 0, 1, 3 & -1 & -2 & -3 & \\
-2 & -1 & 0, 0, 2 & -1, 1, 2 & 0, 2, 3 & -1 & -2 & \\
-1 & 0, -1, 1 & -1, 0, 1 & 0, 1, 1 & -1, 2, 1 & 0, 3, 1 & -1 & \\
-2 & -1 & 0, 0, 0 & -1, 1, 0 & 0, 2, 0 & -1 & -2 & \\
-3 & -2 & -1 & 0, 1, 1 & -1 & -2 & -3 & -4 \\
-4 & -3 & -2 & -1 & -2 & -3 & -4 & -5 \\
-5 & -4 & -3 & -2 & -3 & -4 & -5 & \\
\end{array}
\]

In particular, if we compute \( \widetilde{h}(i, j) \) relative to an \( (n_{ij}, i, j) \)
under \( \text{cl} \) then \( \widetilde{h} \) decreases monotonically with
taxi-cab distance from \( (i, j, i, j) \).

However, if \( (n_{ij}, i, j) \) is above \( \text{cl} \), then \( \widetilde{h} \) defines
a plateau near \( (i, j, i, j) \) where \( \widetilde{h} = h \) and outside
monotonically decreases.
Since we assumed that \( h: \mathbb{Z}^2 \to \mathbb{Z} \) was a Speyer height function, we have \( \lim_{l_i, l_j \to \infty} h(i,j) + l_i + l_j = l_2 \).

Thus, for \( |i-i'| + |j-j'| \) large enough, \( h(i,j) \geq n - |i-i'| - |j-j'| \).

\( \tilde{h} \) agrees with \( h \) for at most a finite region around \( (i_{11}, j_{11}) \) and then monotonically decreases.

Rem: \( \tilde{h} \) is not a height function for this reason but still satisfies:

1. \( \tilde{h}(i,j) = i + j \mod 2 \)
2. \( \tilde{h}(i,j) \pm 1 = \tilde{h}(i \pm 1, j) = \tilde{h}(i, j \pm 1) \).

Speyer calls \( \tilde{h} \) a pseudo-height function.

We now construct \( \tilde{h} \) instead of \( h \)’s.
In general, $\mathcal{G}_h$ looks like:

Definition: An infinite completion of a perfect matching $M$ of $G(n_1, i_1, j_1)$ to an infinite matching $\tilde{M}$ of $\mathcal{G}_h$ so that $\tilde{M}$ uses the diagonal/wrench edges on a co-finite region of $\mathcal{G}_h$, i.e., outside $G(n_1, i_1, j_1)$.

Claim: Face weight $w(M) = w(\tilde{M})$ since the exterior of additional hexagons have 2 edges in $\tilde{M}$ on each face. Treat all faces of $\mathcal{G}_h$ as "closed faces".

Closed faces of $G(n_1, i_1, j_1)$ have weights as expected.

Open faces of $G(n_1, i_1, j_1)$ bordering hexagons have weights off-by-one as in Speyer's formula.
Case where $G(n_1, \tilde{v}_1, \tilde{j}_1) = \emptyset$, i.e., "just on open face".

Since $n_1 = h(\tilde{v}_1, \tilde{j}_1)$, i.e. $(n_1, \tilde{v}_1, \tilde{j}_1) \in \mathcal{L}$.

Case where $G(n_1, \tilde{v}_1, \tilde{j}_1)$ contains a single closed face, i.e.,
$(n_1 - 2, \tilde{v}_1, \tilde{j}_1) \in \mathcal{L}$.
In other words,

**Lemma:** The set of perfect matchings reachable from $\tilde{M} :=$ arbitrary matching of $G(\tilde{y}, \tilde{y}, i)$ completed via including diagonal/wrench edges outside of $G(\tilde{y}, \tilde{y}, i)$

by twisting \[ \begin{array}{c} \square \leftrightarrow \square \end{array} \]
or \[ \begin{array}{c} \square \leftrightarrow \square \end{array} \] etc.

is in bijection with perfect matchings of $G(\tilde{y}, \tilde{y}, i)$ s.t. weights agree.

To prove the Main Theorem it thus suffices to compare "reachable" perfect matchings of $\mathcal{G}_\mathcal{h}$ to $\tilde{\mathcal{G}}_\mathcal{h}$ where

$$h'(i,j) := \begin{cases} h(i,j) + 2 & \text{if } i = i', j = j, \\ h(i,j) & \text{otherwise.} \end{cases}$$

\[ \begin{array}{c} \hline h \rightarrow h^{-1} \rightarrow h \\ h^{-1} \rightarrow h \rightarrow h^{-1} \rightarrow h \hline \end{array} \]
Called "urban renewal" possibly plus

Claim: Changing graph by \( \rightarrow \) to \( \rightarrow \)
does not affect (face)-weighted enumeration of perfect matchings.

Fazes \( F_1, F_2 \) have two fewer edges, one fewer edge in \( M \)
\( F_3, F_4 \) same lengths and some edges of \( M \) incident.
More complicated Claim: Changing graph from

![Graph diagram]

also does not affect (face) weighted enumeration of perfect matchings (with relation $x_6 x'_6 = x_2 x_9 + x_5 x_7$)

**Rem:** Notice in this representative example, faces $F_1$, $F_3$, $F_4$, $F_6$, $F_8$, $F_{10}$ have same number of edges before & after.

Faces $F_2$, $F_5$, $F_7$, $F_9$ have two more edges each (and it is possible to immediately use $\rightarrow\leftarrow\rightarrow$ transform).

**PF of Claim:** Three families of perfect matchings

<table>
<thead>
<tr>
<th>$M_0$</th>
<th>$M'_0$</th>
<th>$M_1$</th>
<th>$M'_1$</th>
<th>$M_2$</th>
<th>$M'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>has no perfect matching edges on center face ($F_6$)</td>
<td>no edge</td>
<td>one edge</td>
<td>one edge</td>
<td>two edges</td>
<td>two edges</td>
</tr>
</tbody>
</table>

Weighted 2-to-1 map $M_2 \rightarrow M_0'$

Weighted 1-to-1 map $M_1 \rightarrow M'_1$

Weighted 1-to-2 map $M_0 \rightarrow M'_2$
If $X'X = WY + VZ$, then

\[
\text{sum of weighted perfect matchings on LHS} = \text{sum of weighted perfect matchings on RHS}
\]
Thus applying the relation $XX' = WY + VZ$ inductively, we can "mutate" height function $f$ until it is in $\mathcal{C}_L$ and $f(n_i, \tilde{i}, \tilde{j}, \tilde{s}) = x_{i,j}$.

Then undoing the sequence of height changes, we find $f(n_i, \tilde{i}, \tilde{j}, \tilde{s})$ when $n_i > h(i, j, s)$.

Need to "mutate" $h'$ until get $\tilde{h}$.

with original $\mathcal{C}_L$. 
Approach via DiFrancesco-Kedem

Let \( U(q, b, c) = \left( \begin{array}{cc} 1 & 0 \\ \frac{c}{b} & \frac{a}{b} \end{array} \right) \), \( V(q, b, c) = \left( \begin{array}{cc} \frac{b}{c} & \frac{a}{c} \\ 0 & 1 \end{array} \right) \)

Claim: \( U_i(q, b, c) V_{i+1}(b, c, d) = V_{i+1}(q, c, d) U_i(q, b, d) \)

\( V_i(q, b, c) U_{i+1}(d, e, f) = U_{i+1}(d, e, f) V_i(q, b, c) \)

\# \( U(q, b, u) V(r, b, c) = V(r, g, b') U(b', g, u) \)

\( \Rightarrow b b' = u v + a c \)

Here \( M = \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right] \)

\( T\)-system (A 2D unrestricted) defined as

\( T_{i,j,k} + T_{i,j,k} = T_{i,j,k} T_{i,j,k} + T_{i,j,k} T_{i,j,k} \)

Technically, get two independent systems depending on parity of \( i+j+k \) mod 2, so enough to focus on one such class like Speyer did.
There is also an $A_1$ case:

\[ T_{ij,k} T_{ij,k-1} = T_{i+j,k} T_{i-j,k+1} \]

an $A_r$ case: (same as $A_{oo}$ case except $i \in \{0,1,\ldots,r\}$ with boundary conditions $T_{0,j,k} = T_{r+j,k} = 1$)

and cases for other Dynkin Diagrams.

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Thm 3.6 of "T-systems, Networks, and Dimeres" by Di Francesco arXiv:1307.0095

For the $A_{oo}$ case with initial conditions

\[ T_{ij,j,k} = t_{ij} \] for some stepped surface $K$

\[ T_{ij,k} = \det M \] with

\[ \left| \begin{array}{cccccc}
    t_{i-j,0} & t_{i-j,1} & \cdots & t_{i-j,l} \\
    t_{i-j+1,0} & t_{i-j+1,1} & \cdots & t_{i-j+1,l} \\
    \vdots & \vdots & \ddots & \vdots \\
    t_{i-j+l,0} & t_{i-j+l,1} & \cdots & t_{i-j+l,l}
  \end{array} \right| 
\]

where shadow $\mathcal{D}$ \(\to\) cone $C(n_j, i_j)$

\[ \mathcal{D}_0 \to \text{open cone } C(n_j, i_j) \]

stepped surface $K \to \mathcal{D}$ defined by height function $h$

Matrix $M$ is a product of $U_i$'s & $V_i$'s depending on heights $K_{Hi}$, $K_{Hi-1}$, $K_{Hi}$, $K_{Hi+1}$.
\[ a \equiv (a, b, d) = U(a, b, d) \]

\[ a \equiv (c, a, b) = V(c, a, b) \]

\[ U(a, b, d) V(c, a, b) = V(c, a, b) U(b, c, d) \]

Let \( M_{\text{go}} \) = non-intersecting lattice paths