Last time we discussed corner invariants $X_i(P)$, where $0 \leq i \leq n-1$ for convex $n$-gon $P$.

Today we coordinatize instead w/ $2n$ real numbers for vertices & sides of $P$.

$$X_i(P) := \begin{cases} \frac{1}{N} \mathcal{X} (\langle \langle i, 4 \rangle \rangle \langle \langle i, i+1 \rangle \rangle \langle \langle i, \overline{i+4} \rangle \rangle) & \text{if } V_i \text{ is a vertex of } P \\ -\mathcal{X} (\langle V_i \cap V_i+4 \rangle \langle V_i \cap V_i+1 \rangle \langle V_i \cap V_i+2 \rangle) & \text{if } V_i \text{ is a side of } P \end{cases}$$

E.g., $P =$ hexagon

$V_6$ is a side $\Rightarrow$ $X_6(P) = -\mathcal{X} (\langle V_6 \cap V_2 \rangle \langle V_5 \cap V_3 \rangle \langle V_6 \cap V_{10} \rangle)$

sides $V_6$ and $V_2$ intersect in a unique pt
and so do $V_6$ and $V_{10}$.

$V_6 \cap V_2$, $V_5 \cap V_3$, $V_6 \cap V_{10}$ all collinear so we define cross-ratio accordingly.
$V_7$ is a vertex $\Rightarrow Y_7(p) = \frac{1}{\lambda(\langle V_7, V_3 \rangle, V_6, V_8, \langle V_3, V_11 \rangle)}$

sides $V_6$ and $V_8$ meet at vertex $V_7$

and $\langle V_7, V_3 \rangle, \langle V_3, V_{11} \rangle$ define two more lines meeting at $V_7$.

We thus define $\lambda(\langle V_7, V_3 \rangle, V_6, V_8, \langle V_3, V_{11} \rangle)$ using the slopes of these 4 lines which all meet at the same point.

OR equivalently, intersect with any $l$ and take cross ratio of the four collinear intersection points.

Call these the $y$-parameters of polygon $P$.

Subscripts taken modulo $2$.

Prop 6.6 of [GRR18] Let $P' = T(P)$, the polygon after applying the pentagram map. For $i = 1, 2, \ldots, 2n$ let $Y'_i$ denote the $y$-parameters of $P'$. Then

$$Y'_i = \begin{cases} 
Y_i^{-1} & \text{if } i \text{ is a side of } P' \quad \text{(a vertex of } P) \\
Y_i \frac{(1+Y_{i-1})(1+Y_{i+1})}{(1+Y_{i-3})(1+Y_{i+3})} & \text{if } i \text{ is a vertex of } P' 
\end{cases}$$
(3) \[ \text{E.g., } P = \text{outside } V_{10} \]

\[ \text{P} \rightarrow T(P) = P' \text{ switches the roles of vertices \& sides.} \]

**Def:** Consider the bipartite quiver \( Q_n \) defined on \( \mathbb{Z}^n \) vertices with arrows around each vertex \( j \) as when \( j \) is odd:

\[ j-3, j-1, j+1, j+3 \]

**E.g.** \( Q_5 = \)

Let \( M_{\text{odd}} = M_{Z_{n-1}} \circ M_{Z_{n-3}} \circ \ldots \circ M_3 \circ M_1 \)

\( \quad M_{\text{even}} = M_{Z_0} \circ M_{Z_2} \circ \ldots \circ M_{Z_4} \circ M_{Z_0} \)

Just like in the case of Zamolodchikov periodicity, \( M_{Z_i} \circ M_{Z_j} \) commute so can rearrange these steps or think of as simultaneous mutation. (Same for constituents of Mod.)
Thm 6.7 of [GR18] (Originally from Glick 2010) Consider the Y-seed \((Y_1, Y_2, \ldots, Y_n, Q_n)\) and apply \(K\) compound mutations \(M_{even} \circ M_{odd} \circ M_{even} \circ \ldots\) alternating between the two.

The resulting Y-seed is \(\left(\left(\frac{Y_{i(1)}}{Y_{i(2)}} \ldots \frac{Y_{i(K)}}{Y_{i(n)}}\right), \left(-1\right)^K Q_n\right)\)

where if we let original \(Q_n\) or its reverse

\(Y_i^c\)'s = Y-parameters of convex \(n\)-gon \(P\)

then \(Y_i^{(K)}\)'s = Y-parameters of \(n\)-gon \(T^K(P)\)

where \(T^K(P)\) is the result after applying the pentagram map \(T\) to \(P\) \(K\) times.

Furthermore, just as cluster variables w/ principal coeff (or in general setting w/ semifield \(P\)) could be rewritten in terms of \(F\)-polynomials, we have the following from Cl. Alg IV

**Prop 3.13** Y-seed components \(Y_{ijt}\) for \(j=1, \ldots, N\) \((N=\#vertices\ in \(Q\))\)

can be rewritten as

\[ Y_{ijt} = \frac{\hat{c}_{ijt}}{Y} \prod_{\substack{c=1 \\text{c-vec}\, j, \text{top or bottom of} \, c \in \tilde{c}_{ijt}}} (\hat{b}_{ijt}) \]  

where

\[ \hat{c}_{ijt} = c\text{-vec} in \, \text{bottom or column } j \text{ in } \tilde{B}_t = \begin{bmatrix} B_{t0} \\ \tilde{c} \end{bmatrix} \]  

assuming \(\hat{b}_{t0} = \begin{bmatrix} b_{t0} \\ 1 \end{bmatrix}\).
The parameters of $T^K(P)$ are given by

$$Y_j(T^K(P)) = \left\{ \begin{array}{ll}
\prod_{i=-k}^{K} F_{j-i} F_{j+i} K \\
\prod_{i=-k+1}^{K} F_{j-i} K \\
\prod_{i=-k}^{K} F_{j-i} F_{j+i} K-1 \\
\prod_{i=-k+1}^{K} F_{j-i} K-1 \\
\end{array} \right. \quad \text{when } j+k \text{ is even}
$$

Taking subscripts of $Y_j$'s modulo $2n$

$$\text{and } F_{j+k} = \sum_{\text{order ideals } \{s_1, s_2, \ldots, s_k\} \in C_k} \prod_{i=1}^{K} Y_{3i+s+i},$$

Instead of giving Glick's definition of posets $P_K$, we will re-express the $F$-polynomials as generating functions of perfect matchings of Aztec Diamonds.

\[ F_{j,1} = 1 + Y_j \]

\[ F_{j,2} = 1 + Y_{j-3} + Y_{j+3} + Y_{j-3} Y_{j+3} + Y_{j-3} Y_{j} Y_{j+3} (1 + Y_{j-1} + Y_{j+1} + Y_{j-1} Y_{j+1}) \]
Based on Elkies-Kuperberg-Larsen-Propp, Max Glick defines the poset $P_k$ to have the vertices

$$\{ (r,s,t) \in \mathbb{Z}^3 \mid 2|s| - (k-2) \leq t \leq (k-2) - 2|s| \text{ and } 2|s| - (k-2) \equiv t \equiv (k-2) - 2|s| \mod 4 \}$$

$$\cup \{ (r,s,t) \in \mathbb{Z}^3 \mid 2|s| - (k-1) \leq t \leq (k-1) - 2|s| \text{ and } 2|s| - (k-1) \equiv t \equiv (k-1) - 2|s| \mod 4 \}$$

with cover relations $(r',s',t') > (r,s,t)$ if $t' = t + 1$ and $|r' - r| + |s' - s| = 1$.

\[\begin{array}{cccc}
t = -2 & t = -1 & \text{Notice} (0,0,-2) \text{ and } (0,0,2) \text{ both in } P_3 \text{ but otherwise } (r,s) \text{ determines } (r,s,t) \text{ uniquely.} \\
\text{Five layers) } & \text{Cr_s(s) values} & \text{illustrated} & \text{ alternatives} \\
\end{array}\]

\[\begin{array}{ccc}
t = 0 & t = 1 & t = 2 \\
\end{array}\]

Superimposing all layers together, we see an Aztec Diamond.

cover relations illustrated
e.g., \( P_4 \) has seven layers

- \( t = -3 \)
- \( t = -2 \)
- \( t = -1 \)
- \( t = 0 \)
- \( t = 1 \)
- \( t = 2 \)
- \( t = 3 \)
order ideals \( \mathbb{D} \) in a poset are subsets that are downward-closed under the cover relation.

Claim: Order ideals of \( \mathbb{D} \) are in bijection with perfect matchings of the Aztec Diamond with height and width of \((2K-1)\) squares.

e.g. \( P_2 \)

\[
\emptyset \quad \{ (-1,0) \} \quad \{ (1,0,-1) \} \quad \{ (-1,0,-1), (1,0,-1) \}
\]

Summing these 8 summands together yields

\[
\{ (-1,0,-1), (1,0,-1), (0,0,0) \} \quad \{ (-1,0,-1), (1,0,-1), (0,0,0), (0,1,1) \}
\]

Recall: weights are \( \prod Y_{3r+5+j} \) over the perfect matchings called height func of the perfect matchings.

Recall: weights are \( \prod Y_{3r+5+j} \) over the perfect matchings called height func of the perfect matchings.

\[\begin{array}{cccc}
& Y_{-3+j} & Y_{3+j} & Y_{-3+j} Y_{3+j} Y_{1+j} Y_{3+j} \\
Y_{-3+j} & Y_{-3+j} Y_{1+j} Y_{3+j} Y_{3+j} & Y_{-3+j} Y_{1+j} Y_{3+j} Y_{3+j} & Y_{-3+j} Y_{1+j} Y_{3+j} Y_{3+j}
\end{array}\]

\[\text{equivalent under:} \quad \begin{array}{ccc}
-3+j & -3+j & 3+j \\
-1+j & -1+j & -1+j
\end{array}\]

These weights called height func of the perfect matchings.

From the bottom of page 5
For $P_3$, we see three squares in the central row yields we then flip the remaining two squares of that row to yield. The flipping the four labeled squares yields

Followed by flipping two squares in the center column:

\[ \text{Flipping the rest of the central column concludes with} \]

Note that the central square is flipped twice, both in the first step and in the last step.

Coincides with $(0, 0 - 2)$ & $(0, 0 + 2)$ both appearing in $P_3$.

Rem: Aztec Diamonds are subgraphs of infinite checkerboard w/ labels.