Proofs from Cl. Alg IV by Fomin-Zelevinsky

Prop 3.9 Let (B, X, Y) be the data associated to seed t and recall $\hat{y}_{j,t} := y_{j,t}\prod_i x_{i,t}^{(b_{ij})_t}$. Then the $\hat{y}_{j,t}$'s mutate as a Y-system, i.e. for $\boxed{t \xrightarrow{\mu_k} t'}$

$$\hat{y}_{j,t'} = \begin{cases} \hat{y}_{k,t}^{-1} & \text{if } j=k \\ \hat{y}_{j,t}\cdot \hat{y}_{k,t}^{(b_{kj})_t} \big/ (\hat{y}_{k,t}+1)^{(b_{kj})_t} & \text{if } j\neq k \ \& \ b_{kj}>0 \\ \hat{y}_{j,t}\cdot\left(\hat{y}_{k,t}^{(b_{kj})_t}+1\right) & \text{if } j\neq k \ \& \ b_{kj}<0 \\ \hat{y}_{j,t} & \text{if } j\neq k \text{ and } b_{kj}=0. \end{cases}$$

Rem: Can write compactly using $[\alpha]_+ := \max(\alpha, 0)$ as

$$\hat{y}_{j,t'} = \begin{cases} \hat{y}_{k,t}^{-1} & \text{if } j=k \\ \hat{y}_{j,t}\cdot y_{k,t}^{[(b_{kj})_t]_+}\,(\hat{y}_{k,t}+1)^{-(b_{kj})_t} & \text{if } j\neq k \end{cases}$$

Pf: $$x_{k,t'} = \frac{y_{k,t}\prod_i x_{i,t}^{[(b_{ik})_t]_+} + \prod_i x_{i,t}^{[-(b_{ik})_t]_+}}{(y_{k,t}\oplus 1)\,x_{k,t}}$$ can be rewritten as

$$x_{k,t'} = \frac{(\hat{y}_{k,t}+1)\prod_i x_{i,t}^{[-(b_{ik})_t]_+}}{(y_{k,t}\oplus 1)\,x_{k,t}}$$

$$y_{k,t'} = y_{k,t}^{-1} \ \& \ \prod_i x_{i,t}^{(b_{ik})_{t'}} = \prod_i x_{i,t}^{-(b_{ik})_{t'}}$$

$\Rightarrow \boxed{\hat{y}_{k,t'} = \hat{y}_{k,t}^{-1} \text{ as desired}}$ since $B_t \xrightarrow{\mu_k} B_{t'}$ includes negating column k.

② $\underline{\text{For } j \neq k}$ $\widehat{y}_{j,t'} = y_{j,t'} \cdot x_{k,t'}^{(b_{kj})_{t'}} \cdot \prod_{i,\, i \neq k} x_{i,t}^{(b_{ij})_{t'}}$

by definition.

Using the above formula for $x_{k,t'}$, we can rewrite this as

$$\widehat{y}_{j,t'} = \frac{y_{j,t'} \left(\widehat{y}_{k,t} + 1\right)^{(b_{kj})_{t'}}}{\left(y_{k,t} \oplus 1\right)^{(b_{kj})_{t'}} x_{k,t}^{(b_{kj})_{t'}}} \prod_{i \neq k} x_{i,t}^{[-(b_{ik})_t]_+ (b_{kj})_{t'} + (b_{ij})_{t'}}$$

Furthermore, exchange matrix mutation can be expressed as

$$(b_{ij})_{t'} = \begin{cases} -(b_{ij})_t & \text{if } i = k \text{ or } j = k \\ (b_{ij})_t + [-(b_{ik})_t]_+ (b_{kj})_t + (b_{ik})_t [(b_{kj})_t]_+ & \text{otherwise} \end{cases}$$

Taking reciprocals (since $(b_{kj})_{t'} = -(b_{kj})_t$), we get

$$\widehat{y}_{j,t'} = \frac{y_{j,t'} \left(y_{k,t} \oplus 1\right)^{(b_{kj})_t} x_{k,t}^{(b_{kj})_t}}{\left(\widehat{y}_{k,t} + 1\right)^{(b_{kj})_t}} \prod_{i \neq k} x_{i,t}^{(b_{ij})_{t'} - [-(b_{ik})_t]_+ (b_{kj})_t}$$

Further, $\underline{\text{exponent}}$ $(b_{ij})_{t'} - [-(b_{ik})_t]_+ (b_{kj})_t$ simplifies to

$$(b_{ij})_t + (b_{ik})_t [(b_{kj})_t]_+$$

We then use $\widehat{y}_{j,t} = y_{j,t} \cdot \prod_{i} x_{i,t}^{(b_{ij})_t}$ by def^n to see

and $y_{j,t'} = y_{j,t}\, y_{k,t}^{[(b_{kj})_t]_+} \left(y_{k,t} \oplus 1\right)^{-(b_{kj})_t}$

$$\widehat{y}_{j,t'} = \frac{y_{j,t}\, y_{k,t}^{[(b_{kj})_t]_+} x_{k,t}^{(b_{kj})_t}}{\left(\widehat{y}_{k,t} + 1\right)^{(b_{kj})_t}} \prod_{i \neq k} x_{i,t}^{(b_{ij})_t + (b_{ik})_t [(b_{kj}}$$

(for $j \neq k$)

(3) $\Rightarrow$ $$\widehat{y}_{j,t'} = \frac{\widehat{y}_{j,t}\, y_{k,t}^{[(b_{kj})_t]_+} \prod_{i \neq k} x_{i,t}^{(b_{ik})_t [(b_{kj})_t]_+}}{(\widehat{y}_{k,t} + 1)^{(b_{kj})_t}}$$

[since $\widehat{y}_{j,t} = y_{j,t} \cdot x_{k,t}^{(b_{kj})_t} \cdot \prod_{i \neq k} x_{i,t}^{(b_{ij})_t}$]

and we use $\widehat{y}_{k,t} = y_{k,t} \cdot \prod_{i \neq k} x_i^{(b_{ik})_t}$ [since $b_{kk} = 0$]

to obtain

$$\widehat{y}_{j,t'} = \frac{\widehat{y}_{j,t}\, \widehat{y}_{k,t}^{[(b_{kj})_t]_+}}{(\widehat{y}_{k,t} + 1)^{(b_{kj})_t}}$$ as desired. $\blacksquare$

Thm 3.7 Let A be a cluster algebra defined over a semifield $\mathbb{P}$ (so that $\mathbb{X}$ and $\mathbb{Y}$ mutate w/ $\oplus$ in the formula as above). Then $x_{i,t}$ can be written in terms of $x_i = x_{i,t_0}$'s & $y_i = y_{i,t_0}$'s as

$$x_{i,t} = \frac{\overset{L_{i,t}}{\text{Laurent Poly in } \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n]}}{F_{i,t}|_{\mathbb{P}}(y_1, \ldots, y_n)}$$

where $F_{i,t}$ (F-poly) $:= L_{i,t}[1, \ldots, 1, y_1, \ldots, y_n]$
and $F_{i,t}|_{\mathbb{P}}$ replaces $+$ w/ $\oplus$ from $\mathbb{P}$.

PF by induction on length of the sequence of mutations from t_0 to t.

Let $Y_{i,t}$ denote $\widehat{y}_{i,t}|_{x_1 = \cdots = x_n = 1}$ so that $Y_{i,t}$'s mutate like a Y-seed but w/ $+$ instead of $\oplus$.

④ Prop 3.12 $$F_{k,t'} = \frac{(Y_{k,t}+1)\prod_i F_{i,t}^{[-(b_{ik})_t]_+}}{(Y_{k,t}\oplus 1)\, F_{k,t}} \quad \text{if } \boxed{t \xrightarrow{\mu_k} t'}$$

PF: Analogue of formula for $x_{k,t'}$ but letting $\underset{x_{1,t_0}}{x_1} = \dots = \underset{x_{n,t_0}}{x_n} = 1$.

Continuing the proof by induction, assume

$$x_{i,t} = \frac{L_{i,t}(x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1, \dots, y_n)}{F_{i,t}|_{\mathbb{P}}(y_1, \dots, y_n)} \qquad \forall\, i \text{ \& seeds } t \ \ell \text{ steps from } t_0.$$

We wish to show we have such an expression for $x_{k,t'}$ where $\boxed{t \xrightarrow{\mu_k} t'}$.

we can rewrite $x_{k,t'} = \dfrac{(\hat{Y}_{k,t}+1)\prod_i x_{i,t}^{[-(b_{ik})_t]_+}}{(Y_{k,t}\oplus 1)\, x_{k,t}}$ as

$$x_{k,t'} = \frac{(\hat{Y}_{k,t}+1)\prod_i \left(\dfrac{L_{i,t}}{F_{i,t}|_{\mathbb{P}}}\right)^{[-(b_{ik})_t]_+}}{(Y_{k,t}\oplus 1)\left(\dfrac{L_{k,t}}{F_{k,t}|_{\mathbb{P}}}\right)}$$

$$= \frac{\dfrac{(\hat{Y}_{k,t}+1)\prod_i L_{i,t}^{[-(b_{ik})_t]_+}}{L_{k,t}}}{(Y_{k,t}\oplus 1)\,\dfrac{\prod_i F_{i,t}^{[-(b_{ik})_t]_+}|_{\mathbb{P}}}{F_{k,t}|_{\mathbb{P}}}}$$

We have separated $+$ and $\oplus$ at this point, and we now use Prop 3.12 to finish the proof.

(5) Letting $x_1 = \dots = x_n = 1$, we recover $L_{i,t} \to F_{i,t}$

Thus $\hat{y}_{i,t} \to Y_{i,t}$

$$x_{k,t'}\Big|_{x_1=\dots=x_n=1} = \frac{\left[\dfrac{(Y_{k,t}+1)\prod_i F_{i,t}^{[-(b_{ik})_t]_+}}{(Y_{k,t} \oplus 1)\qquad F_{k,t}}\right]}{\left[\dfrac{\prod_i F_{i,t}^{[-(b_{ik})_t]_+}\big|_{\mathbb{P}}}{F_{k,t}|_{\mathbb{P}}}\right]}$$

$$= F_{k,t'} / F_{k,t'}|_{\mathbb{P}} \quad \left[\text{using } (Y_{k,t}+1)|_{\mathbb{P}} = Y_{k,t} \oplus 1\right] \quad \blacksquare$$

<u>Rem</u>: We omitted the proof of the Laurent Phenomenon, i.e. the fact that $\prod_i L_{i,t}^{[-(b_{ik})_t]_+} / L_{k,t}$ is again a Laurent poly.

<u>Cor 6.3</u> We can rewrite $x_{i,t} = \left(\dfrac{L_{i,t}}{F_{i,t}|_{\mathbb{P}}}\right)$ where $F_{i,t} := L_{i,t}[1,1,\dots,1,y_1,y_2,\dots,y_n]$ as

$$x_{i,t} = \frac{F_{i,t}[\hat{y}_{1,t_0}, \hat{y}_{2,t_0}, \dots, \hat{y}_{n,t_0}]\, x_1^{(g_{1,i})_t} \cdots x_n^{(g_{n,i})_t}}{F_{i,t}|_{\mathbb{P}}[y_1, y_2, \dots, y_n]}$$

where $\vec{g}_{i,t} = \begin{bmatrix} (g_{1i})_t \\ (g_{2i})_t \\ \vdots \\ (g_{ni})_t \end{bmatrix}$ is the <u>g-vector</u> associated to elt i of seed t.

6 Proof of Corollary 6.3: We first define the following $\mathbb{Z}^n$-grading on Laurent monomials in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n]$ by

$$x_i \mapsto \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} i\text{th} \quad \& \quad y_j \mapsto \begin{bmatrix} -(b_{1j})_{t_0} \\ -(b_{2j})_{t_0} \\ \vdots \\ -(b_{nj})_{t_0} \end{bmatrix}$$

and extend linearly.

$$\text{Thus } \deg(\hat{y}_{j,t_0}) = \deg\left(y_j \cdot \prod_i x_i^{(b_{ij})_{t_0}} \right) = \vec{0} \quad \forall\, 1 \le j \le n.$$

Based on the mutation rules for $\hat{y}_{j,t} \to \hat{y}_{j,t'}$, i.e. invert, multiply by $(\hat{y}_{k,t}+1)^{\pm}$ or $\hat{y}_{k,t}$, it follows that $\boxed{\deg(\hat{y}_{j,t}) = \vec{0}}$ $\forall\, 1 \le j \le n$ and seeds t.

Prop: For any cluster variable $x_{i,t}$, all contained Laurent monomials, as summands, have the same $\mathbb{Z}^n$-grading.

To see this, it is trivially true for the base cases of $x_{i,t_0} = x_i$. Then any other cluster variable is built up by

$$x_{k,t'} = \frac{(\hat{y}_{k,t}+1) \prod_i x_{i,t}^{[-(b_{ik})_t]_+}}{(y_{k,t} \oplus 1)\, x_{k,t}}.$$

By induction, every Laurent monomials in $x_{i,t}$ & $x_{k,t}$ have the same $\mathbb{Z}^n$-gradings, and $\deg(\hat{y}_{k,t}) = \deg(1) = \vec{0}$.

⑦ Hence, we define $\vec{g}_{i;t}$ as the $\mathbb{Z}^n$-grading for all Laurent monomials of cluster variable $x_{i;t}$, initialized as $\vec{g}_{i;t_0} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ith $\forall\ 1 \le i \le n$. (see Cor 6.2 of [Cl. Alg IV])

Hence, consider the Laurent polynomial $L_{i;t} \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1, \dots, y_n]$ and replace x_i w/ $\lambda_i x_i$

and y_j w/ $\lambda_1^{-(b_{1j})_{t_0}} \lambda_2^{-(b_{2j})_{t_0}} \cdots \lambda_n^{-(b_{nj})_{t_0}} y_j$

It follows that $\underline{L_{i;t}^{\lambda} = \lambda_1^{(g_{1;i})_t} \lambda_2^{(g_{2;i})_t} \cdots \lambda_n^{(g_{n;i})_t} L_{i;t}}$

(w/ the above replacement)

Letting $\lambda_j = x_j^{-1}\ \forall\ 1 \le j \le n$, we get

$$L_{i;t}^{\lambda} = L_{i;t}[1, 1, \dots, 1, \hat{y}_{1,t_0}, \dots, \hat{y}_{n,t_0}] = \frac{L_{i;t}}{x_1^{(g_{1;i})_t} x_2^{(g_{2;i})_t} \cdots x_n^{(g_{n;i})_t}}$$

LHS $= F_{i;t}(\hat{y}_{1,t_0}, \dots, \hat{y}_{n,t_0})$ and we obtain from numerator of the RHS

$$\boxed{L_{i;t} = x_1^{(g_{1;i})_t} x_2^{(g_{2;i})_t} \cdots x_n^{(g_{n;i})_t} F_{i;t}(\hat{y}_{1,t_0}, \dots, \hat{y}_{n,t_0})}$$

Finally, $x_{i;t} = \dfrac{L_{i;t}}{F_{i;t}|_{\mathbb{P}}}$ finishes the proof.

(8) Using a dual of g-vectors, we can similarly obtain a Compact formula for the Y-system (as rational functions).

See Props 5.1 and 5.6 of [Cl. Alg. IV]: Given

$B_{t_0} = [b_{ij}]_{t_0}$, $\tilde{B}_{t_0} = \left[\frac{B_{t_0}}{I}\right]$, $2n$-by-n and suppose

$t_0 \xrightarrow{\mu_{i_1}\circ\mu_{i_2}\circ\cdots\circ\mu_{i_\ell}} t$ and $\tilde{B}_t = \left[\frac{B_t}{C_t}\right]$, i.e. $C_{t_0} = I$.

we define the c-vector $\vec{c}_{j,t} = \begin{bmatrix} (c_{1j})_t \\ \vdots \\ (c_{nj})_t \end{bmatrix}$

By the definition of exchange matrix mutation (if $t \xrightarrow{\mu_k} t'$),

$$\vec{c}_{j,t'} = \begin{cases} -\vec{c}_{j,t} & \text{if } j = k \\ \vec{c}_{j,t} + [b_{kj}]_+ \, \vec{c}_{k,t} & \text{if } (c_{ik})_t \geq 0 \ \forall i \\ \vec{c}_{j,t} - [-b_{kj}]_- \, \vec{c}_{k,t} & \text{if } (c_{ik})_t \leq 0 \ \forall i \end{cases}$$

Comparing w/ the formula in Prop 3.9,

we see

$$\widehat{Y_{j,t}}\Big|_{\mathrm{Trop}} \quad \left[\begin{array}{l}\text{i.e. replacing } + \text{ w/ } \oplus \text{ where} \\ \mathbb{P} \text{ is tropical semifield w/} \\ y_1^{a_1}\cdots y_n^{a_n} \oplus 1 = 1 \ \forall\, a_1,\dots,a_n \geq 0\end{array}\right]$$

and letting $x_1 = \cdots = x_n = 1$

$$\| \quad y_1^{(c_{1j})_t} \cdots y_n^{(c_{nj})_t}.$$

sign-coherence assumption that is proven via rep. theory of quivers with potentials or via scattering diagrams

(9) i.e. $\hat{Y}_{j,t}|_{x_1=\dots=x_n=1} = Y_{j,t}$ $\left[\begin{array}{c}\text{rational function} \\ \text{w/ } +\end{array}\right]$

and $\hat{Y}_{j,t}|_{x_1=\dots=x_n=1}|_{Trop} = Y_{j,t}|_{Trop} = \vec{y}^{\,\vec{c}_{j,t}}$

As a consequence, Prop 3.13: We can rewrite

$$\boxed{Y_{j,t} = \vec{y}^{\,\vec{c}_{j,t}} \cdot \prod_{i=1}^{n} F_{i,t}^{(b_{ij})_t}}.$$

PF: $\frac{\hat{Y}_{j,t}|_{x_1=\dots=x_n=1}}{Y_{j,t}} = \hat{Y}_{j,t}|_{x_1=\dots=x_n=1} = y_{j,t}|\cancel{} \cdot \prod_{i=1}^{n} x_{i,t}^{(b_{ij})_t}\Big/_{x_1=\dots=x_n=1}$

$$\Rightarrow Y_{j,t} = y_{j,t}|\cancel{} \cdot \prod_{i=1}^{n} F_{i,t}^{(b_{ij})_t}$$

and under the same assumption as sign-coherence,
all F-polynomials $F_{i,t}$ have a constant term of 1

and hence $F_{i,t}|_{Trop} = 1$ (since $F_{i,t}$ is a Polynomial in $\oplus$).

$$\Rightarrow Y_{j,t}|_{Trop} = y_{j,t}|_{Trop} \cdot \prod_{i=1}^{n} \left(F_{i,t}|_{Trop}\right)^{(b_{ij})_t}$$

$$= y_{j,t} \Rightarrow \boxed{Y_{j,t}|_{Trop} = \vec{y}^{\,\vec{c}_{j,t}}}$$

and finishes the proof.

(10) Lastly, assuming sign-coherence, we can prove

$$\underline{G_t^T = C_t^{-1}}$$ ($T \leftarrow$ transpose) relating g-vectors & c-vectors for seed t associated to a quiver.

Rem: skew-symmetrizable case involves slight technical change.

See Nakanishi-Zelevinsky "On Tropical Dualities in Cl. Algs."

Pf omitted

We also get a variant definition of g-vectors assuming sign-coherence or equiv. F-polys have constant term 1

$\vec{g}_{j,t}$:= exponent vector of the unique Laurent monomial left in $x_{j,t}\big|_{y_1=\cdots=y_n=0}$ where

$x_{j,t}$ defined by $\tilde{B}_{t_0} = \begin{bmatrix} B \\ I \end{bmatrix}$, e.g. principal coeffs,

or equiv. $x_{j,t}$ defined w/ $\mathbb{P}$ = Tropical semifield w/ $y_1 = u_1^1, \ldots, y_n = u_n^1$.