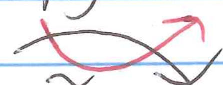
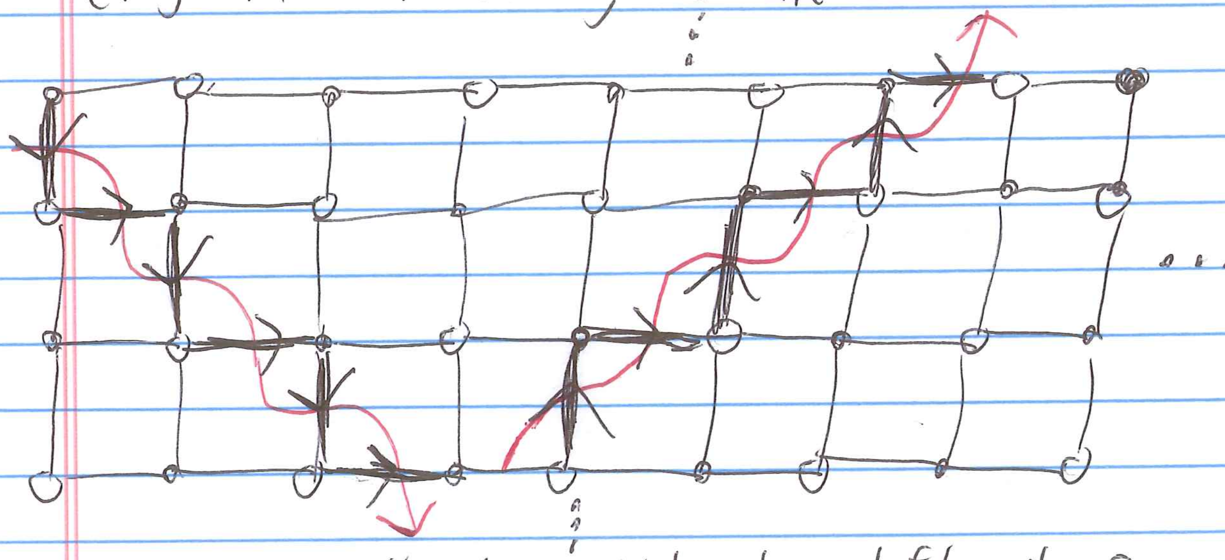
11/30/18

Using the construction from [GK11], as discussed in 11/28/18 notes, the <u>bipartite graph on a torus Γ_Δ</u>, built from polygon Δ, is a <u>minimal admissible graph</u> in the following sense.

<u>Def 2.1 of [GK11]</u> On surface S, bipartite graph Γ is <u>minimal</u> if its <u>alternating strands</u> (a variant of zig-zag paths) exhibit • no loops, • no self intersections, and • no parallel bigons  when drawn on the universal cover $\tilde{\Gamma}$ of $\tilde{S}$.

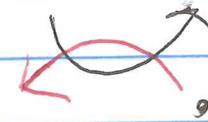
(E.g. if S is a torus, $\tilde{S} = \mathbb{R}^2$ is its universal cover).



Zig-zag paths turn right at •, left at ∘.

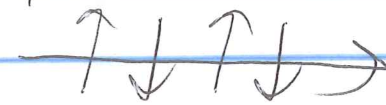
Alternating strands traverse midpoints of $\tilde{\Gamma}$'s edges, counter-clockwise around •, clockwise around ∘.

<u>Rem:</u> Recording which edges are utilized/traversed along the zig-zag-path/alternating strand yields a bijection.

<u>Note:</u> In the def'n of <u>minimal</u>, <u>antiparallel bigons are OK</u> .

②

Note: Meaning of admissible previously discussed:

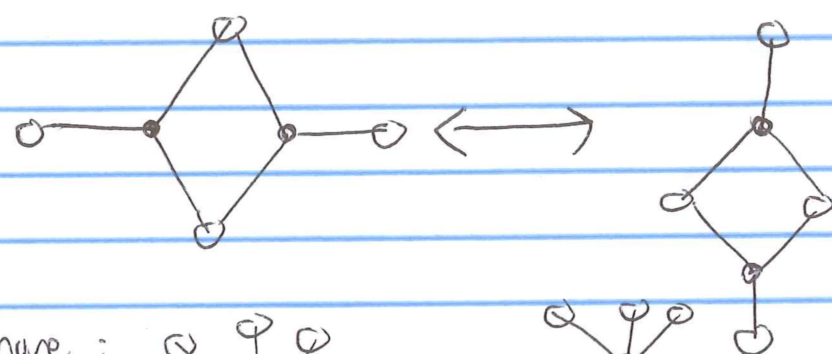
- going along a loop $\gamma_{\vec{i}}$, we see 

- minimal # of intersections.

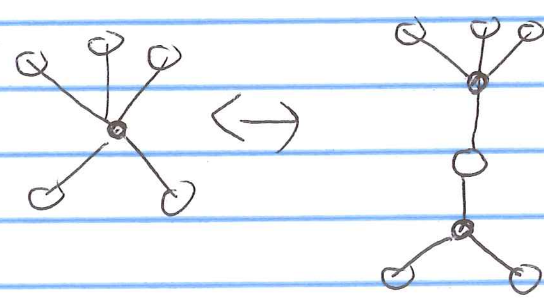
Thm 2.5 of [GK11] $\Delta \to \Gamma_\Delta$ is a minimal admissible graph on a torus, and any two Γ_1, Γ_2 (minimal admissible graphs) constructed from Δ are related by a sequence of
- spider moves

and or
- valence 2 vertex moves

spider move:

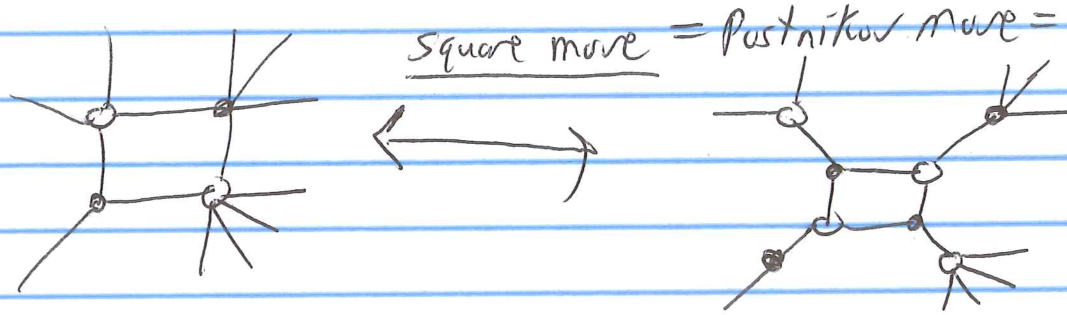


valence 2 vertex move:

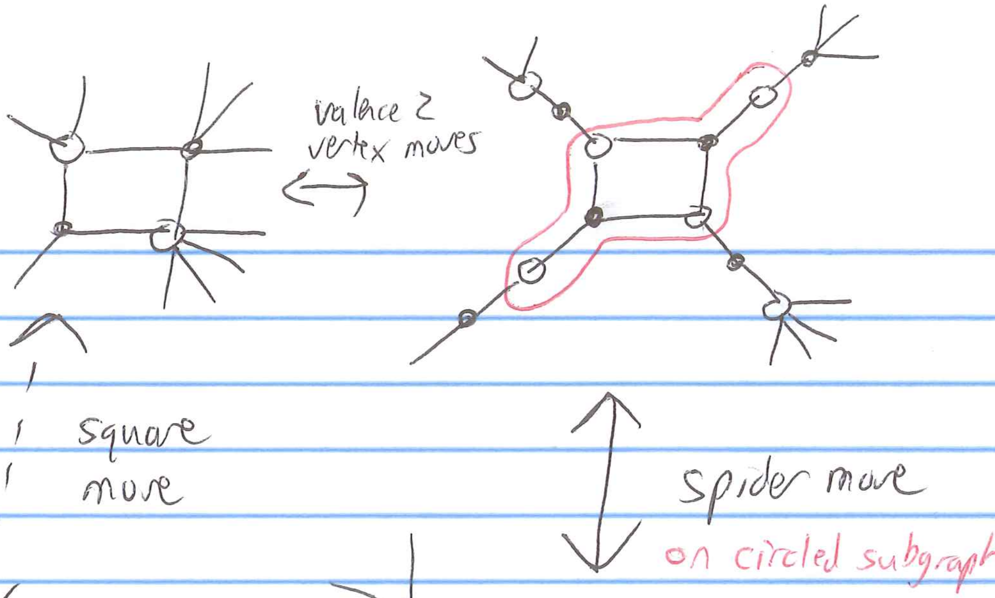


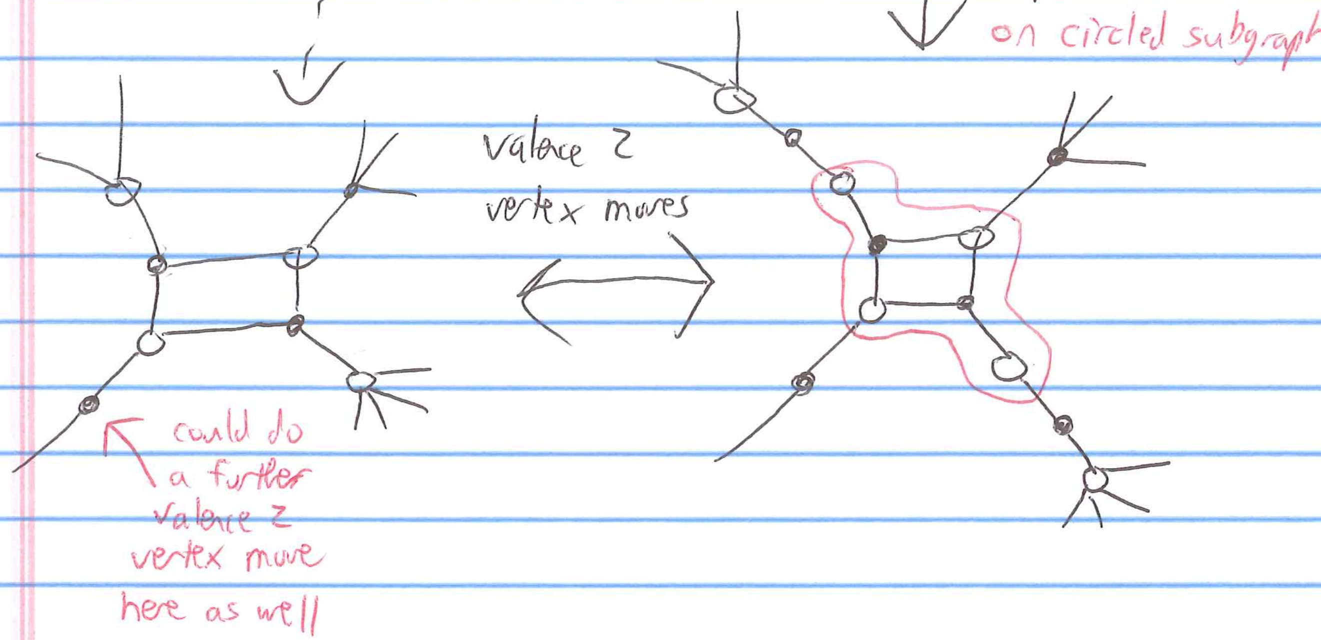
Rem: Can also combine these together

square move = Postnikov move = urban Renewal

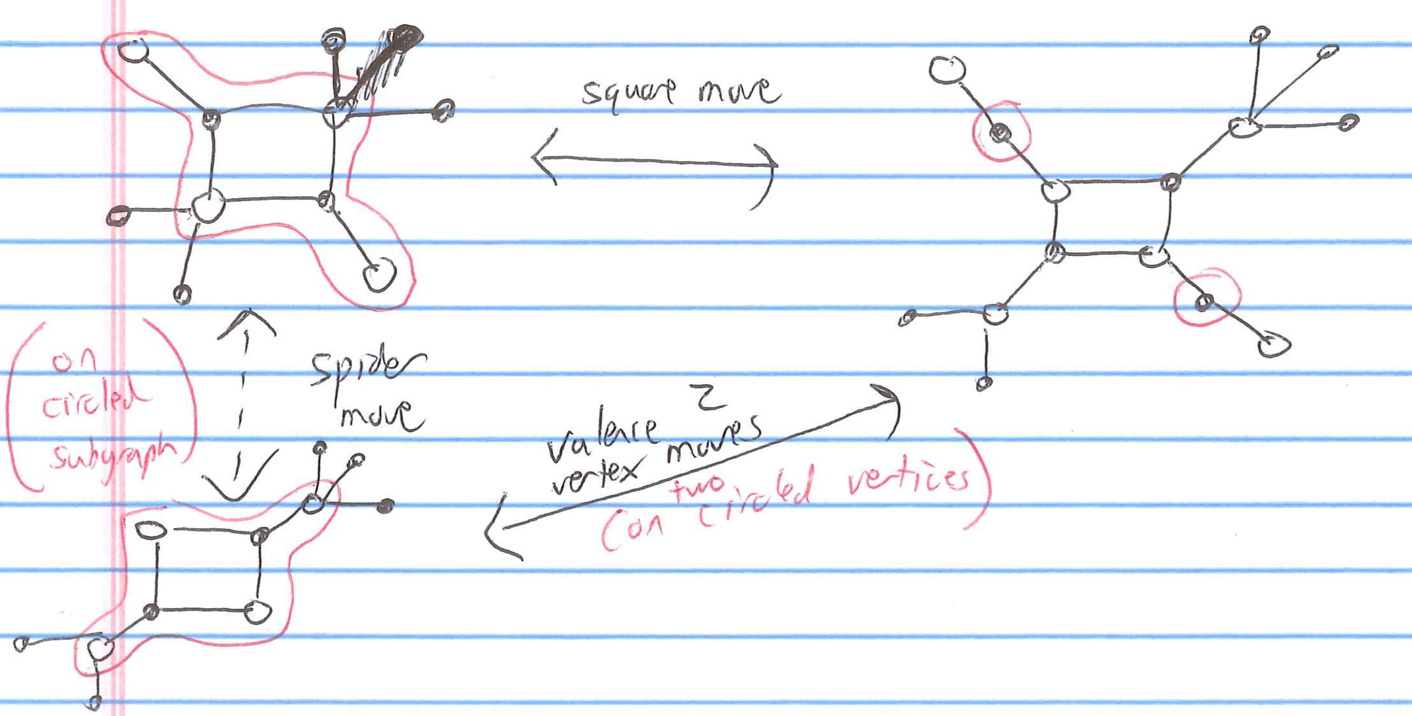


(3) In Particular,

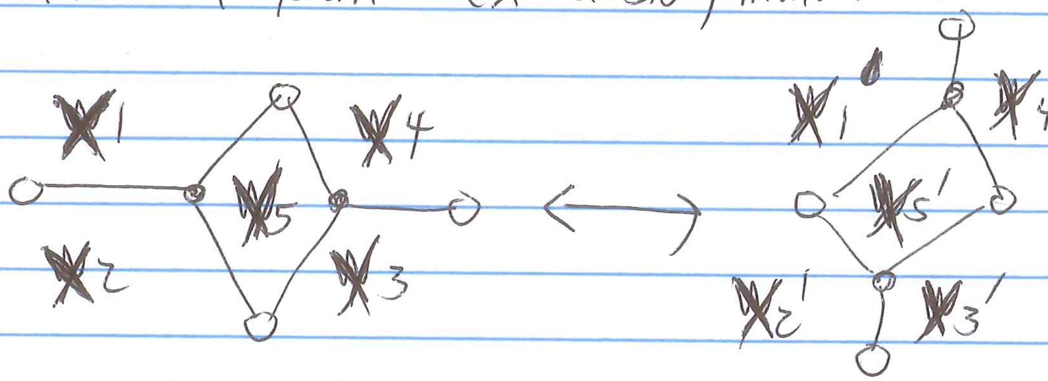




OR can see spider move as a special case of the other two:



(4) **Thm 4.7 [GK11]** Spider Moves affect face weights like Y-system (X-cluster) mutation:



$$X_5' = X_5^{-1}$$

$$X_1' = X_1(1+X_5) \qquad [\text{same for } X_3']$$

$$X_2' = X_2(1+X_5^{-1})^{-1} \qquad [\text{same for } X_4']$$

Furthermore: The Poisson structure is preserved under spider moves + weight changes

$\{X_i', X_j'\} = \{X_i, X_j\}$ if we have a bijection between faces F_i's & F_i''s.

(*) Induces a discrete cluster integrable system by iterating such mutations.

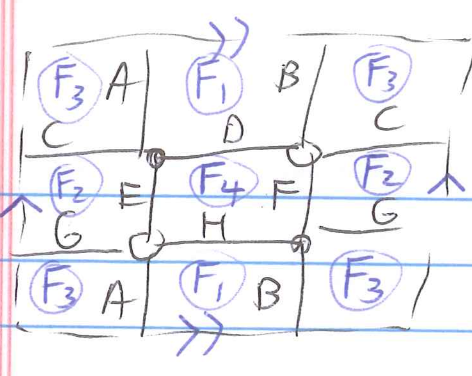
Casimirs, i.e. conserved quantities of the Poisson structure correspond to

Zig-zag paths in Γ or equivalently,

ratios of perfect matchings corresponding to two adjacent exterior vertices of Δ. ($B-1$ independent Casimirs).

Example

(5)



$$K = \begin{bmatrix} -Cz_1^{-1} + D & Az_2^{-1} + E \\ Bz_2 + F & Gz_1 - H \end{bmatrix}$$

$$\det K = -CG - AB - DH - EF$$
$$+ CHz_1^{-1} + DGz_1 - AFz_2^{-1} - BEz_2$$

$$w(F_1) = \frac{AB}{DH},\quad w(F_2) = \frac{EF}{CG},\quad w(F_3) = \frac{CG}{AB},\quad w(F_4) = \frac{DH}{EF}$$

(orient faces F_i clockwise & •→○ in numerator, ○→• in denominator)

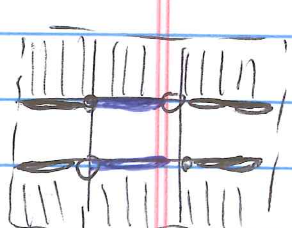
If we choose a reference matching, e.g. $\underline{M_0 = DH}$

then remaining three matchings which are coeffs of $z_1^0 z_2^0$ can be obtained from M_0 by multiplying by $w(F_i)$'s:

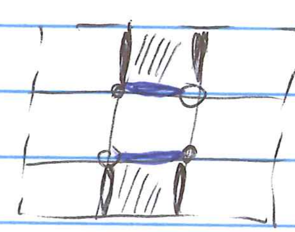
$$CG + AB + DH + EF = DH\left[X_1X_3 + X_1 + 1 + X_1X_2X_3\right]$$

where we let $\underline{X_i = w(F_i)}$

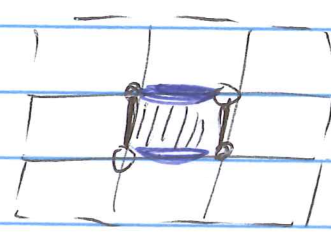
Combinatorially, alternating products obtained from M/M_0 are cycles around faces (exactly agreeing w/ this algebra).



$\dfrac{CG}{DH}$



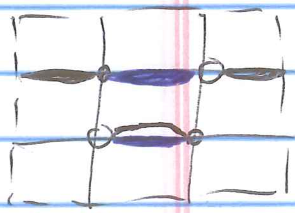
$\dfrac{AB}{DH}$



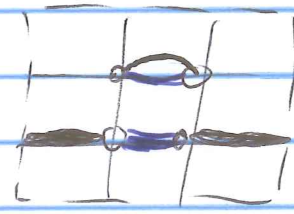
$\dfrac{EF}{DH}$

$X_4^{-1} = X_1X_2X_3$

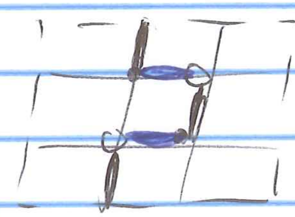
⑥ The other four M/M_0 look like



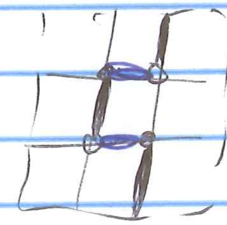
$\frac{CH}{DH}$



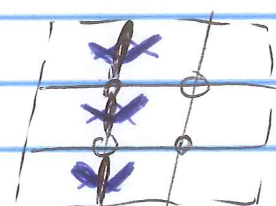
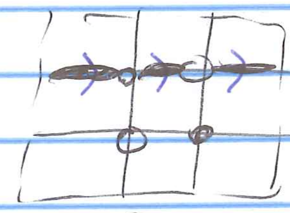
$\frac{DG}{DH}$



$\frac{AF}{DH}$



$\frac{BE}{DH}$

If we let $z_2 =$  $\frac{E}{A}$ $\qquad z_1 =$  $\frac{D}{C}$

we indeed have

$$z_1^{-1}, \quad z_1 X_1 X_3, \quad z_2^{-1} X_4^{-1} = z_2^{-1} X_1 X_2 X_3, \quad z_2^{+1} X_1$$

$$\frac{C}{D} \qquad \frac{G}{H} = \frac{D}{C} \cdot \frac{AB}{DH} \cdot \frac{CG}{AB} \qquad \frac{AF}{DH} = \frac{A}{E} \cdot \frac{EF}{DH} \qquad \frac{BE}{DH} = \frac{E}{A} \cdot \frac{AB}{DH}$$

<u>Claim:</u> Corresponding X-Cluster Poisson Variety has

Hamiltonian $H = 1 + X_1 + X_1 X_3 + X_1 X_2 X_3$

and Casimirs (From ratios) $C_1 = \frac{z_1^{-1}}{z_2^{-1} X_1 X_2 X_3}$, $C_2 = \frac{z_1 X_1 X_3}{z_2 X_1}$, $C_3 = \frac{z_2^{-1} X_1 X_2 X_3}{z_1 X_1 X_3}$, $C_4 = \frac{z_2 X_1}{z_1^{-1}}$.

(7)

Three out of Four of these Casimirs are independent.

$C_1 = z_1^{-1} z_2 X_4, \quad C_2 = z_1 z_2^{-1} X_3, \quad C_3 = z_1^{-1} z_2^{-1} X_2,$

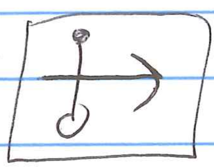
$C_4 = z_1 z_2 X_1$ (we use $X_4 = X_1^{-1} X_2^{-1} X_3^{-1}$)

By Leibniz Rule, iteratively, $\boxed{\{X_1^{a_1} X_2^{a_2} \cdots X_K^{a_K}, Y\} = (a_1\varepsilon_1 + a_2\varepsilon_2 + \dots + a_K\varepsilon_K)\, X_1^{a_1} X_2^{a_2} \cdots X_K^{a_K}\, Y}$ under the assumptions $a_i \in \mathbb{Z}$ and each $\{X_i, Y\} = \varepsilon_i X_i Y$.

ε-matrix for this Γ given by

$$\begin{array}{c|cccc|cc} x_1 & 0 & 0 & 2 & -2 & -1 & 1 \\ x_2 & 0 & 0 & -2 & 2 & 1 & -1 \\ x_3 & -2 & 2 & 0 & 0 & 1 & 1 \\ x_4 & 2 & -2 & 0 & 0 & -1 & -1 \\ \hline z_1 & 1 & -1 & -1 & 1 & 0 & -1 \\ z_2 & -1 & 1 & -1 & 1 & 1 & 0 \end{array}$$

with quiver $Q_\Gamma =$
$$\begin{tikzcd} 1 \arrow[r, Rightarrow] & 3 \arrow[d, Rightarrow] \\ 4 \arrow[u, Rightarrow] & 2 \arrow[l, Rightarrow] \end{tikzcd}$$

using 

We get $\varepsilon(z_i, X_j)$ by noting:

edge D of z_1 borders faces F_1, F_4

edge C of z_1 borders faces F_2, F_3

while edge A of z_2 borders F_1, F_3

edge E of z_2 borders F_2, F_4.

(8) Using this ε-matrix, we indeed see for $C_1 = z_1^{-1} z_2 x_4$,

$$\{C_1, \gamma_i\} = -\varepsilon(z_1, \gamma_i) + \varepsilon(z_2, \gamma_i) + \varepsilon(x_4, \gamma_i)$$

$$= \begin{cases} -(+1) + (-1) + (2) & \text{if } \gamma_i = x_1 \\ -(-1) + (+1) + (-2) & \text{if } \gamma_i = x_2 \\ -(-1) + (-1) + 0 & \text{if } \gamma_i = x_3 \\ -(+1) + (+1) + 0 & \text{if } \gamma_i = x_4 \\ -(0) + (+1) + (-1) & \text{if } \gamma_i = z_1 \\ -(1) + (0) + (1) & \text{if } \gamma_i = z_2 \end{cases}$$

$\Rightarrow$ $\{C_1, \gamma_i\} = 0$ for any γ_i.

similarly $\{C_2, \gamma_i\} = \{C_3, \gamma_i\} = \{C_4, \gamma_i\} = 0$

so C_1, C_2, C_3, C_4 are indeed Casimirs.

Hamiltonian $H = 1 + x_1 + x_1 x_3 + x_1 x_2 x_3$

(Corresponding to the unique internal lattice point in this case)

yields a vector field and a flow

(*) continuous integrable system for symplectic structure mod the Casimirs.

(9) As in Prop 3.15 & Sec 3.7 & Thm 3.12 of [GK11]

$S = \text{area}(\Delta) \in \mathbb{Z}/2$, $I = \#$ interior pts (Δ),
$B = \#$ boundary pts (Δ),

Pick's Thm $\Rightarrow$

$2I + B - 2 = 2S$

& $B - 1 = \#$ independent Casimirs = dimension of the center of Poisson alg corresponding to Γ.

$2S = |\text{Faces}(\Gamma)|$ and $\dim \mathcal{L}_\Gamma = |\text{Faces}(\Gamma)| + 1$
$= 2S + 1$

$\Rightarrow 2I = (2S+1) - (B-1)$

$= \dim \mathcal{L}_\Gamma - \dim(\text{center Poisson Alg})$

$\Rightarrow 2I = \dim$ (symplectic structure induced by Γ)

$\Rightarrow I = \frac{1}{2}$(symplectic dim)

$\Rightarrow$ yields an integrable system if $I = \#$ indep. Hamiltonians.

Indeed that is the number of Independent Hamiltonians (that commute w/ each other).

In E.g. $\dim \mathcal{L}_\Gamma = |\text{Faces}(\Gamma)| + 1 = 5$, # ind Casimirs $= 3$
1 Hamiltonian for $5-3=2$-dimensional symplectic structure,