Lecture Math 8680: Poisson brackets compatible with cluster algebra structures

Def: A Poisson algebra is a commutative associative algebra equipped w/ a Poisson bracket

\[ \{ y, z \} : g \times g \rightarrow g \] which is a skew-symmetric bilinear map

\[
\begin{align*}
\{ax+by, cz+dz\} &= ac \{ x, z \} + ad \{ x, z \} + bc \{ y, z \} + bd \{ y, z \} \\
\text{and} \quad \{ y, x \} &= -\{ x, y \}
\end{align*}
\]

satisfying the Leibniz identity

\[ \{ F_1 F_2, F_3 \} = F_1 \{ F_2, F_3 \} + \{ F_1, F_3 \} F_2 \]

and the Jacobi identity

\[ \{ F_1, \{ F_2, F_3 \} \} + \{ F_2, \{ F_3, F_1 \} \} + \{ F_3, \{ F_1, F_2 \} \} = 0. \]

Rem: If we let g = C^\infty(M), the algebra of smooth functions on a symplectic manifold, the usual product & chain rules of differentiation imply these identities.
Rem: Given an associative algebra $A$, let $[x, y]$ be defined as the commutator

$$[x, y] = xy - yx.$$ This yields a Lie algebra with Lie bracket $[x, y]$ and together with the multiplicative structure of $A$, this is a Poisson algebra.

Rem: The tensor algebra of a Lie algebra is a Poisson algebra.

Def: Given a cluster algebra $A$, we say a Poisson bracket $\{x, y\}$ on $A$ is compatible with the cluster algebra structure if every cluster $X$ of $A$ is log-canonical w.r.t. $\{x, y\}$, i.e.,

$$\exists \text{ skew-symmetric matrix } N_X \text{ w/ entries } N_i^j \text{ s.t. } \{x_i, x_j\} = N_i^j x_i x_j \forall x_i, x_j \in X.$$ Equivalently, $\{\log x_i, \log x_j\} = N_i^j x_i x_j x_i x_j$.

Lemma: Let $\tilde{B}$ be the extended exchange matrix for a cluster algebra, such that $\tilde{B}$ is $(m+n)$-by-$n$ and rank $n$.

Let $N_X$ be the $(m+n)$-by-$(m+n)$ matrix assoc. to a compatible Poisson bracket.

Then $\tilde{B}^T N_X = \mathbf{E} D O J$ where $D = n$-by-$n$ diagonal w/ nonzero entries.
Construction \[ \text{Muller '16} \] For a cluster algebra from an un punctured surface \( \mathcal{A} \) such that \( \widehat{B} \) has full rank, \( y \) can build a compatible Poisson structure \( \text{Via} \)

\[ n_{ij} := (\# \text{endpts of } T_i \& T_j \text{ s.t. } T_j \text{ clockwise from } T_i \left[ \text{not nec. immediately} \right]) \]

\[ - (\# \text{endpts of } T_i \& T_j \text{ s.t. } T_j \text{ counter-clockwise from } T_i) \]

Example: \( \begin{array}{c}
\text{4} \\
\text{3} \\
\text{1} \\
\text{2} \\
\text{5} \\
\text{6} \\
\text{7}
\end{array} \) yields \( \widehat{B} = \begin{bmatrix}
-1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 0 \\
-1 & 1 \\
0 & -1
\end{bmatrix} \)

and \( n_X = \begin{bmatrix}
0 & -1 & 1 & -1 & 0 \\
1 & 0 & -1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 & -1
\end{bmatrix} \)

Note:

\( \widehat{B} n_X = \begin{bmatrix}
40 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0
\end{bmatrix} \)
Can find other Poisson structures as well, an entire subspace of choices.

Another one appears in Example 4.1 of [GSV10]:

\[
\begin{bmatrix}
0&1&-1&-1&1&1&0 \\
-1&0&-1&-1&0&1&-1 \\
1&1&0&1&2&2&1 \\
1&1&-1&0&1&2&0 \\
-1&0&-2&-1&0&1&0 \\
-1&-1&-2&-1&0&1&0 \\
0&1&-1&0&0&1&0
\end{bmatrix}
\]

\[
\beta J_{X_2} = \begin{bmatrix} -20 \\ 0 \\ 0 \end{bmatrix}.
\]

In particular, \( x_1' = x_2 x_5 + x_4 x_6 \) and we wish to show \( S_{e_j} \) also log-canoninal on the neighboring cluster \( X' = \{ x_1', x_2, x_3, \ldots, x_7 \} \), i.e. \( \exists \lambda \in \mathbb{R} \) s.t.

\[
\lambda x_1 x_i = \lambda x_2 x_5 x_i - \lambda x_4 x_6 x_i.
\]

For \( i = 2, 3, \ldots, 7 \) using \( J_{X_2} \) or \( J_{X_2}^0 \).

First, more generally observe that \( \sum_{j} \frac{1}{a_j} b_j = -\frac{1}{a} \sum_{j} a_j b_j \).

Essentially from \( \frac{d}{dx}(1/x) = -\frac{1}{x^2} \) and chain rule, but we will use Leibniz identity.
pf: Note that \( \sum_{j} b_j = 0 \) for all \( b \) since \( 1 \) is a constant function.

Thus, \( \sum_{0}^{1} a_j \cdot b_j = \frac{1}{a} \sum_{0}^{1} a \cdot b_j + \frac{1}{a} \sum_{0}^{1} b_j \)

\( \Rightarrow a \sum_{0}^{1} b_j = -\frac{1}{a} \sum_{0}^{1} a \cdot b_j \Rightarrow \sum_{0}^{1} b_j = -\frac{1}{a^2} \sum_{0}^{1} a \cdot b_j \).

In the special case \( \sum_{0}^{1} b_j = \Lambda \cdot a \cdot b \) (Clay-canonical),

then \( \sum_{0}^{1} b_j = -\frac{\Lambda \cdot a \cdot b}{a} \).

We now compute \( \sum_{1}^{1} x_j \cdot x_i \):

\[
\sum_{1}^{1} x_j \cdot x_i = \left( \begin{array}{l}
\text{Letting, for } j, k \in \{1, 2, 3\}, \\
\sum_{1}^{1} x_j \cdot x_k = \lambda_{j k} x_j \cdot x_k, \\
w/ \lambda_{j k} = \lambda \end{array} \right) \\
\sum_{1}^{1} x_j \cdot x_i + \sum_{1}^{1} x_j \cdot x_i
\]

\( = x_2 \left( \sum_{1}^{1} x_j \cdot x_i \right) + \sum_{1}^{1} x_j \cdot x_i + \sum_{1}^{1} x_j \cdot x_i \)

\( = x_2 \left( \sum_{1}^{1} x_j \cdot x_i \right) + x_2 \left( \sum_{1}^{1} x_j \cdot x_i \right) + x_2 \left( \sum_{1}^{1} x_j \cdot x_i \right) \)

\( = -\lambda \cdot \frac{x_2 x_5 x_i}{x_1} + \lambda _{2 5} \cdot \frac{x_2 x_5 x_i}{x_1} + \lambda _{2 6} \cdot \frac{x_2 x_5 x_i}{x_1} \)

\( + \frac{x_2 x_6 x_i}{x_1} \)
Here

\[ Sx'_i x_i = \lambda'_i x'_i x_i \quad \text{if} \]

\[-\lambda'_i + \lambda_{5i} + \lambda_{2i} = -\lambda_{1i} + \lambda_{6i} + \lambda_{4i} = \lambda'_i \]

In fact, looking at both of these \( S \)-matrices, we see that

\[ \text{sum of 2nd & 5th rows} = \text{sum of 4th & 6th rows} \]

(except in first column)

This calculation also helps us define how \( S \)-matrix changes under mutation:

- replace entries of first row & column
  \[ \lambda_{1i} \rightarrow \lambda_{1'i} \text{, the common value defined above} \]
  \[
  \text{and leave } \lambda_{j'k} \text{ the same otherwise (for } j, k \neq 1)\]
Rem: Having a full rank $\tilde{B}$ is key. See Example 4.2 of [GSV10].

If $\tilde{B} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ (Which has rank 2 not 3)

$x_1' = \frac{x_2 + x_3}{x_1}$ and a Poisson compatible structure would require

$$\{x_1', x_2\} = \left\{\frac{x_2}{x_1}, x_2\right\} + \left\{\frac{x_3}{x_1}, x_2\right\}$$

$$= \frac{1}{x_1}\left(x_2, x_2\right) + x_2\left(\frac{1}{x_1}, x_2\right) + \frac{1}{x_1}\left(x_3, x_2\right) + x_3\left(\frac{1}{x_1}, x_2\right)$$

$$\Rightarrow \lambda_1^2 = \lambda_1 \left(\frac{x_2}{x_1}\right) - \lambda_2 \left(\frac{x_2}{x_1} + \frac{x_3}{x_1}\right) = \lambda_1 \left(\frac{x_2}{x_1}\right) - \lambda_2 \left(\frac{x_2 + x_3}{x_1}\right)$$

$$\Rightarrow \lambda_2 = \lambda_1 = -\lambda_2 - \lambda_3 = 0$$

By compatibility of we can similarly use $\{x_2', x_3\}$ to conclude

we'd need $\lambda_1 \neq \lambda_2$ and thus only

Poisson compatible structure is trivial in this case.