9/28/18 **Math 8680**: Today we dig deeper into the theory of Poisson structures as related to cluster algebras, focusing on the case of cluster algebras from Grassmannians.

We begin by looking at the restriction that the extended exchange matrix $\tilde{B}$ have <u>full rank</u>.

To do this, we start w/ the following Lemma.

<u>Lemma 3.23</u> If rank $\tilde{B} = n$ and $\tilde{B}' = \mu_k \tilde{B}$ then $\tilde{B}'$ has rank n also.

<u>Pf</u>: If the lth column of $\tilde{B}$ contains entry b_{kl} s.t. $b_{kl} < 0$, then consider

$$\tilde{B} = \tilde{B} - b_{kl}(\text{column } l),$$ repeating this for each such column.

Similarly, if the jth row of the resulting $\tilde{B}$ contains entry $b_{jk} > 0$, then we redefine

$$\tilde{B} = \tilde{B} + b_{jk}(\text{row } j)$$ repeating this for each such row.

Finally, we finish by multiplying the kth rows and kth cols by -1.

All of these operations are row & column operations & yields $\tilde{B}''$.

(2) Since $\widetilde{B}'$ can be obtained only by row & column operations, <u>rank $\widetilde{B}'$ = rank $\widetilde{B}$.</u>

Consequently, the rank of the exchange matrix assoc. to a seed of a cluster algebra is an <u>invariant under mutation.</u>

Furthermore because of this descriptions via row and column operations, we can write

$$\widetilde{B}' = \mu_k \widetilde{B} \quad \text{as} \quad E_\varepsilon \widetilde{B} F_\varepsilon, \text{ i.e. matrix multiplication,}$$

where
$$E_\varepsilon = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & 0 \\ & 0 & & -1 & & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

where kth column is
$$\begin{bmatrix} [-\varepsilon \cdot b_{1k}] \\ [-\varepsilon \cdot b_{2k}] \\ \vdots \\ -1 \\ \vdots \\ [-\varepsilon \cdot b_{n+m\,k}] \end{bmatrix}$$
& $\varepsilon = \pm 1$ is a chosen constant.

F_ε defined similarly except kth row is $\left[[\varepsilon \cdot b_{k1}] \; \varepsilon \cdot b_{k2} \cdots -1 \cdots \varepsilon \cdot b_{kn} \right]$ w/ the same choice of ε.

This description of mutation will be important when we get to quantum cluster algebras in a few weeks.

We continue with our discussion of compatible Poisson structures. (Unlike text we will assume matrices have more rows than cols and will assume $\tilde{B}$ has $n \times n$ B on top which is skew-symmetric)

Definition

We define τ-coordinates as the products $\boxed{\tau_j = x_1^{\beta_{1j}} x_2^{\beta_{2j}} \cdots x_{n+m}^{\beta_{n+m,j}}}$ where

$$\vec{\beta} = \begin{bmatrix} \beta_{1j} \\ \beta_{2j} \\ \vdots \\ \beta_{n+m,j} \end{bmatrix}$$

is the jth column of the deformed matrix

$$\beta = \hat{B} + K \quad \text{defined as}$$

$$\beta = \left[\begin{array}{c|c} B & -C^T \\ \hline C & \begin{matrix} x_{n+1} & & 0 \\ & \ddots & \\ 0 & & x_{n+m} \end{matrix} \end{array}\right] \quad \text{if} \quad \tilde{B} = \left[\begin{array}{c} B \\ \hline C \end{array}\right].$$

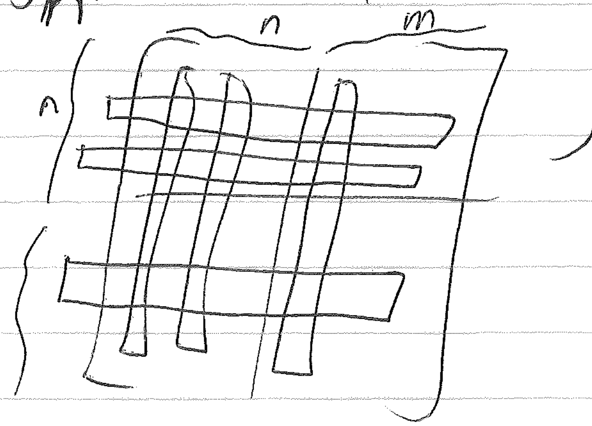
Here $x_{n+1}, \ldots, x_{n+m}$ are integers we treat as indeterminates.

Lemma 4.3 There exists a choice of $x_{n+i} \in \mathbb{Z}$ such that the map $\{x_1, \ldots, x_{n+m}\} \longmapsto \{\tau_1, \ldots, \tau_{n+m}\}$ is nondegenerate $\Longleftrightarrow$ rank $\tilde{B} = n$ (full rank).

Pf: If rank $\tilde{B} < n$ then rank $(\hat{B} + K) < n+m$ $\Rightarrow$ map to τ-coordinates is degenerate (i.e. $\det(\hat{B}+K) = 0$).

So assume rank $\tilde{B} = n$, and for generality, assume rank $B = K \leq n$.

(4) By selecting j_1th, j_2nd, $\dots$, j_nth rows & cols of $\widehat{B+K}$ simulateneously

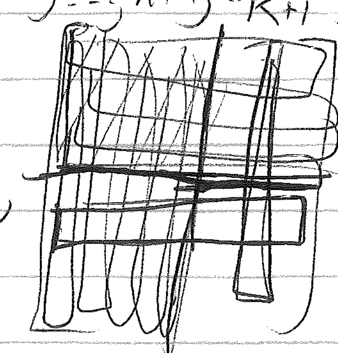


can build a nonzero $n \times n$ minor Q
(since rank $\tilde{B} = n$)

For $j_i > n$, let corresponding $x_{j_i} = 1$
and leave the others outside minor Q as indeterminates common x.

Then $\det Q$ = polynomial in x whose leading coefficient equals

$(2n-k)\times(2n-k)$ minor using 1st, 2nd, $\dots$, nth, j_{k+1}st, $\dots$, j_nth rows and columns.



Applying simultaneous row & col operations, we can turn this minor into

$$\left(\begin{array}{cc|c} B_1 & 0 & 0 \\ 0 & 0 & B_3 \\ \hline 0 & B_4 & B_5 \end{array}\right)$$

where B_1 $k \times k$ = first k rows & cols of $\tilde{B}$

$\begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$ equiv to B $(n \times n)$ because rank $B = k$.

Yields $\det = \det B_1 B_3 B_4$ but rank $\tilde{B} = n$
$\Rightarrow \det B_1 \cdot \det B_3 \neq 0$ & similarly $\det B_4 \neq 0$
by skew-symmetrizability.

(5)

$\Rightarrow$ leading ~~coe~~ coefficient of $\det(\hat{B}+K) \neq 0$.

$\Rightarrow$ $\boxed{\text{rank } \tilde{B} = n \text{ implies } \tilde{X} \mapsto \tilde{\tau} \text{ non-deg. for some choice of } x\text{'s.}}$

This non-degenerate change of coordinates yields a new set of log-canonical coords w.r.t. the Poisson bracket $\{\cdot,\cdot\}$

(Assuming commutativity of mult)

Observe: $\{abc, def\} = a\{bc, def\} + \{a, def\}bc$

$= ab\{c, def\} + a\{b, def\}c + \{a, def\}bc$

$= abde\{c, f\} + abd\{c, e\}f + ab\{c, d\}ef + \ldots$

$\Rightarrow$ In general, $\{x_1 \cdots x_n, y_1 \cdots y_m\} = \sum_{k=1}^{n} \sum_{l=1}^{m} x_1 \cdots \hat{x}_k \cdots x_n y_1 \cdots \hat{y}_l \cdots y_m \{x_k, y_l\}$

and if we assume $\{x_k, y_l\} = \lambda_{kl} x_k y_l \Rightarrow$

$\{x_1 \cdots x_n, y_1 \cdots y_m\} = \left(\sum_{k=1}^{n} \sum_{l=1}^{m} \lambda_{kl} \right) (x_1 \cdots x_n) \cdot (y_1 \cdots y_m)$.

We can adapt this if we have exponents or even negative exponents accordingly, hence

$\boxed{\Omega^{\tau} = (\hat{B}+K)\, \Omega^{\tilde{x}}\, (\hat{B}+K)^T}$ where

$\hat{B}+K$ was the nondegenerate change-of-coordinates matrix.