10/1/18 Math 8680 $(n+m)\times n$

Recall from last time: Given $\tilde{B} = \left[\begin{array}{c} B \\ \hline C \end{array}\right]$,

we define $\widehat{B+K} = \left[\begin{array}{c|c} B & -C^T \\ \hline C & \begin{matrix} x_{n+1} & & 0 \\ & \ddots & \\ 0 & & x_{n+m} \end{matrix} \end{array}\right]$,

$\tau_j = \prod_{i=1}^{n+m} x_i^{\beta_{ij}}$ where exponents are $j$th column. (Laurent monomial)

$\tilde{X} = \{x_1, \dots, x_{n+m}\} \xrightarrow{(\widehat{B+K})^T} \{\tau_1, \dots, \tau_{n+m}\}$ non-degenerate for some choice of $x_i$'s

$\iff \text{rank } \tilde{B} = n$.

Then $\Omega^{\tau} = (\widehat{B+K})\, \Omega^{\tilde{X}}\, (\widehat{B+K})$.

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(part of) Thm 4.5 If $\tilde{B}$ has rank $n$ & $\nexists$ permutation matrix $P$ s.t. $PBP^T$ block-diagonal, i.e. $B$ is irreducible, then

the first $n$ columns of $\Omega^{\tau}$, which we denote as $\Omega^{\tau}[n+m; n]$ satisfies

$$\boxed{\Omega^{\tau}[n+m; n] = \text{scalar multiple of } \tilde{B}}.$$

PF: Suppose $\{x_i, x_j\} = w_{ij} x_i x_j$ for $i,j \in \{1, 2, \dots, n+m\}$.
After mutation in the $i$th direction,

$$\{x_i', x_j\} = \left\{ \frac{1}{x_i} \left( \prod_{b_{ki}>0} x_k^{b_{ki}} + \prod_{b_{ki}<0} x_k^{-b_{ki}} \right), x_j \right\}$$

(2)

$$= \frac{x_j}{x_i} \prod_{b_{ki}>0} x_k^{b_{ki}} \Big( \sum_{b_{ki}>0} b_{ki} w_{kj} - w_{ij} \Big)$$

$$+ \frac{x_j}{x_i} \prod_{b_{ki}<0} x_k^{b_{ki}} \Big( - \sum_{b_{ki}<0} b_{ki} w_{kj} - w_{ij} \Big)$$

Thus the new cluster is compatible w/ the Poisson bracket $\Leftrightarrow$

$$\boxed{\sum_{b_{ki}>0} b_{ki} w_{kj} - w_{ij} = \sum_{b_{ki}<0} b_{ki} w_{kj} - w_{ij} \quad \forall j \neq i.}$$

Hence $\{x_i', x_j\} = \lambda_{ij} x_i' x_j$ under this condition.

$$\Rightarrow (\hat{B} + K)^T \Omega^{\tilde{x}}[n+m; n] = \tilde{B}^T \Omega^{\tilde{x}} = [\overset{n}{D} \ \overset{m}{0}]$$

where $D$ is diagonal. ($DB = \Omega^{\tau}[n,n]$ is skew-symmetric)

and $\Omega^{\tau}$ chosen so $\Omega^{\tau}[n+m; n]$ proportional to $\tilde{B}$.
(when $B$ skew-symmetric and irreducible)

Note: Technical changes if $B$ block diagonal after simultaneous permutation of rows & cols, or if $B$ skew-symmetrizable.

(Rem: From above equation, we see that $\{\cdot,\cdot\}$ compatible w/ cluster structure equivalent to $\tilde{B}^T \Omega^{\tilde{x}} = [D \ 0]$.)

③

**Example 4.6** Let $\tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ \hline 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$,

$\Omega^{\tilde{x}} = \begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 1 & 0 \\ -1 & 0 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & -1 & 0 & 1 & 2 & 0 \\ -1 & 0 & -2 & -1 & 0 & 1 & 0 \\ -1 & -1 & -2 & -2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}$, $\hat{B}+K = \begin{bmatrix} 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & x & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & x & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & x & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & x \end{bmatrix}$

$$\Omega^{\tau} = (\hat{B}+K)^T \Omega^{\tilde{x}} (\hat{B}+K) = \begin{bmatrix} 0 & 2 & 0 & -2 & 2 & -2 & 0 \\ -2 & 0 & 2 & 0 & 0 & 2 & -2 \\ 0 & -2 & 0 & 4 & 0 & 2 & 3 \\ 2 & 0 & -4 & 0 & 1 & 2 & 2 \\ -2 & 0 & 0 & -1 & 0 & 0 & -1 \\ 2 & -2 & -2 & -2 & 0 & 0 & 0 \\ 0 & 2 & -3 & -2 & 1 & 0 & 0 \end{bmatrix}$$

Notice: $\Omega^{\tau}[7;2] = 2 \cdot \tilde{B}$.
(First 2 columns)

$\tau_i$'s are Laurent monomials

$\tau_1 = \dfrac{x_2 x_5}{x_4 x_6}$, $\tau_2 = \dfrac{x_3 x_6}{x_1 x_7}$, $\tau_3 = \dfrac{x_3}{x_2}$, $\tau_4 = x_1 x_4$

$\tau_5 = \dfrac{x_5}{x_1}$, $\tau_6 = \dfrac{x_1 x_6}{x_2}$, $\tau_7 = x_2 x_7$.

(4) We now consider how $\tau$-coordinates transform under mutation.

Lemma 4.4 If $\{x_1, \dots, x_{n+m}\} \xmapsto{(\widehat{B+K})^T} \{\tau_1, \dots, \tau_{n+m}\}$

After mutation in $i$th direction (w/ $B'$ = new exchange matrix)

$$\{x_1, \dots, x_i', \dots, x_{n+m}\} \xmapsto{(\widehat{B'}+K)^T} \{\tau_1', \dots, \tau_{n+m}'\}$$

Then $\tau_i' = \dfrac{1}{\tau_i}$

$$\tau_j' = \begin{cases} \tau_j \left(\frac{1}{\tau_i}+1\right)^{-b_{ij}} & \text{for } b_{ij} > 0 \\ \tau_j (\tau_i+1)^{-b_{ij}} & \text{for } b_{ij} < 0 \\ \tau_j & \text{for } b_{ij} = 0 \ \& \ i \neq j \end{cases}$$

Pf: straight-forward calculation using the definition of matrix mutation & $(\widehat{B}+K)^T$.

As a consequence, a compatible poisson bracket on $\widetilde{X} = \{x_1, \dots, x_{n+m}\}$ can be identified w/ a skew-symmetric bilinear form $w$ on the $(n+m)$-dim'l vector space $L$ spanned by $\log \tau_i$'s.

$$w(\log \tau_i, \log \tau_j) = \{\log \tau_i, \log \tau_j\} = w_{ij} \in \mathbb{Z}.$$

This skew-symm, bilinear form $w$ turns $L$ into a <u>symplectic vector space</u>.

we use $\log(Fg) = \log F + \log g$, $\log\left(\frac{1}{F}\right) = -\log F$,

and $\dfrac{\partial \log F}{\partial F} = \dfrac{1}{F}$

We let $\langle e_1, \dots, e_{n+m} \rangle$ be a basis for $L$
where $e_i := \log \tau_i$

Then extended mutation matrix $\widehat{B}$ (notice no $K$ here) allows us to interpret mutation in $j$th direction as

$$\langle e_1, \dots, e_{n+m} \rangle \longmapsto \langle e_1', \dots, e_{n+m}' \rangle \quad \text{where}$$

$$e_i' = \begin{cases} -e_j & \text{if } i = j \\ e_i & \text{if } w(e_j, e_i) \geq 0 \ \& \ i \neq j \\ \sigma_{e_j}(e_i) & \text{if } w(e_j, e_i) < 0 \end{cases}$$

where $\sigma_v : L \to L$ is a <u>symplectic transvection</u>, the map $x \mapsto x - w(x, v) \cdot v$.

Hence we interpret a cluster alg corresponding to a specific <u>lattice & symplectic vec space $L$</u>, and a seed is <u>a choice of coordinates / basis</u> with <u>mutation</u> being a transition to different coordinates/basis & hence changes $w$-form

(6) Notice $\log(\tau_i') = \log\left(\frac{1}{\tau_i}\right) = -\log \tau_i$

$$e_i' = -e_i \quad \boxed{\checkmark}$$

For $j \neq i$, $\log(\tau_j') = \log(\tau_j) - b_{ij} \log\left(1 + \frac{1}{\tau_i}\right)$

$$= \log(\tau_j) - b_{ij} \log(1+\tau_i) + b_{ij} \log(\tau_i) \quad \text{for } b_{ij} > 0$$

and $\log(\tau_j') = \log(\tau_j) - b_{ij} \log(1+\tau_i)$ for $b_{ij} < 0$

Notice that if we think of $\tau_i$ as "large",

$$\log \tau_i \approx \log(1+\tau_i)$$

Thus for $j \neq i$, $e_j' \approx e_j$ for $b_{ij} > 0$

& $e_j' \approx e_j - b_{ij} e_i$ for $b_{ij} < 0$

agreeing w/ def'n of $\{e_1', \ldots, e_{n+m}'\}$'s. ▨

Remark: Can have this symplectic vector space $L$ only if $n+m$ is even.

Need to handle $n+m$ odd differently, we discuss pre-symplectic structures next.