10/15/18 (1)

**Review:** We have seen how

- $\lambda$-lengths change like cl. vars under flips/mutations
- how $\lambda$-lengths & cross-ratio coordinates related

**Today:**
- How do cross-ratio coordinates change under flips/mutations?
- If we consider extended matrix $\tilde{B}$ by adding frozen variables w/ arbitrary rows, how to encode mutations geometrically?
- What is algebraic relationship between how these two coordinate systems change under mutation?
- Returning to symplectic & Poisson geometric interpretations

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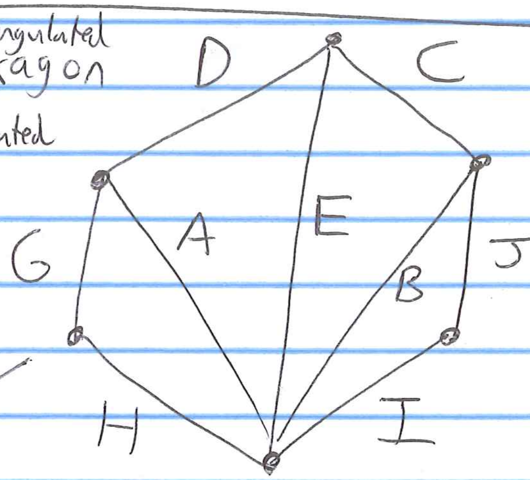
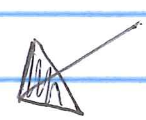
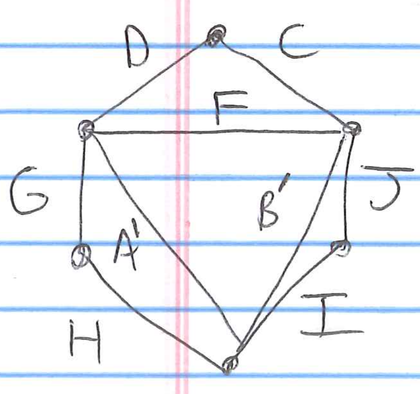
Consider the triangulated hexagon inscribing usual triangulated quadrilateral.



We flip $E \to F$




$\boxed{\text{Choosing convention w/o negative sign}}$

Using $x(E) = \dfrac{\lambda(A)\lambda(C)}{\lambda(B)\lambda(D)}$,

let us compute $x(F) = x(E')$ and $x(A')$, $x(B')$.

② **Warm-up:** $$\overset{x(E')}{\overset{\|}{x(F)}} = \frac{\lambda(B')\lambda(D)}{\lambda(A')\lambda(C)} = \frac{\lambda(B)\lambda(D)}{\lambda(A)\lambda(C)}$$

$$\Rightarrow \boxed{x(E') = x(E)^{-1}}$$

Notice by keeping same orientation

$$x(A) = \frac{\lambda(H)\lambda(D)}{\lambda(G)\lambda(E)} \quad \text{while} \quad x(A') = \frac{\lambda(H)\lambda(F)}{\lambda(G)\lambda(B)}$$

Using Ptolemy, we can rewrite

$$x(A') = \frac{\lambda(H)\left[\dfrac{\lambda(A)\lambda(C)+\lambda(B)\lambda(D)}{\lambda(E)}\right]}{\lambda(G)\lambda(B)}$$

$$= \frac{\lambda(H)\lambda(A)\lambda(C)}{\lambda(G)\lambda(B)\lambda(E)} + \frac{\lambda(H)\cancel{\lambda(B)}\lambda(D)}{\lambda(G)\cancel{\lambda(B)}\lambda(E)}$$

$$\Rightarrow \boxed{x(A') = x(A)\cdot\left[x(E)+1\right]}$$

Analogously, $x(B) = \dfrac{\lambda(E)\lambda(J)}{\lambda(C)\lambda(I)}$, $x(B') = \dfrac{\lambda(A)\lambda(J)}{\lambda(F)\lambda(I)}$

we rewrite $\lambda(F)$ using Ptolemy again, and get

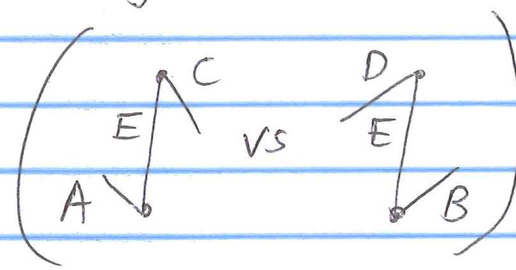
$$\boxed{x(B') = x(B)\cdot\left[x(E)^{-1}+1\right]^{-1}} = \left[x(B)^{-1} + x(B)^{-1}x(E)\right]^{-1}$$

(3) Enlarging the hexagon to an octagon, we see

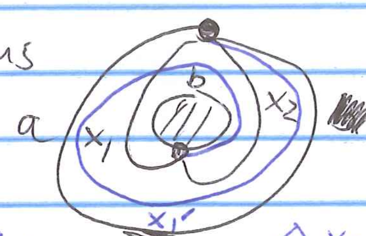
$$x(C') = x(C)\cdot\left[x(E)+1\right]$$

and $$x(D') = x(D)\cdot\left[x(E)^{-1}+1\right]^{-1}$$ similarly

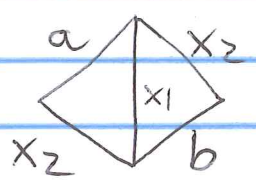
i.e. RHS dependent on relative orientation between arc $\alpha$ and mutated arc $E$



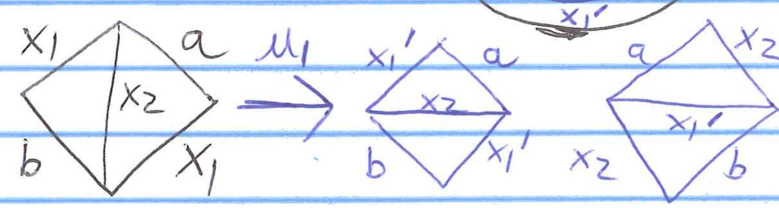

**Remark**: For a surface like annulus




where we have ideal quadrilaterals




and




$$x(x_1) = \frac{\lambda(x_2)^2}{\lambda(a)\lambda(b)}, \quad x(x_2) = \frac{\lambda(a)\lambda(b)}{\lambda(x_1)^2}$$

and it follows after mutation $\mu_1$ (resp. $\mu_2$)

PF: Exercise

$$x(x_1') = x(x_1)^{-1} \quad \left(\text{resp. } x(x_1)\left[x(x_2)^{-1}+1\right]^{-2}\right)$$

$$x(x_2') = x(x_2)\left[x(x_1)+1\right]^2 \quad \left(\text{resp. } x(x_2)^{-1}\right)$$

$\overset{\shortparallel}{x_2}$

exchange matrix

So has the form $$x(\alpha') = \begin{cases} x(\alpha)\cdot\left(x(E)^{\pm}+1\right)^{b_{\alpha E}} & \text{for } \alpha \neq E \\ x(\alpha)^{-1} & \text{for } \alpha = E \end{cases}$$

where $\pm$ determined by sign of $b_{\alpha E}$.

(4) Return now to $\tau$-coordinates (Lemma 4.4 from 10/1/18 notes or sec 4.1 of [GSV10])

Cross-ratio coords transform exactly like $\tau$-coordinates when

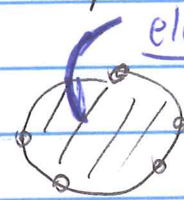
$\tilde{B}$ is $(n+c)\times n$ w/ extended cluster given by triangulation $T$ plus boundary segments.

$\tau_i' = \tau_i^{-1}$ and $\tau_j' = \tau_j(1+\tau_i)^{+b_{ij}}$ for $b_{ij}>0$, $\tau_j' = \tau_j(1+\tau_i^{-1})^{b_{ij}}$ for $b_{ij}<0$

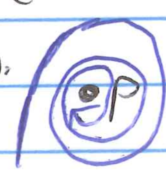
$\tau_j' = \tau_j$ if $b_{ij}=0$

We now build up different coordinates that we will relate to $\tilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}$ for arbitrary $C$.

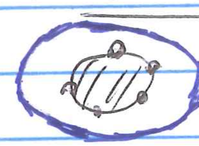
Def: An elementary lamination is like an arc except instead of connecting two marked points, its endpoints either

- intersect $\partial S \setminus M$ (e.g. )

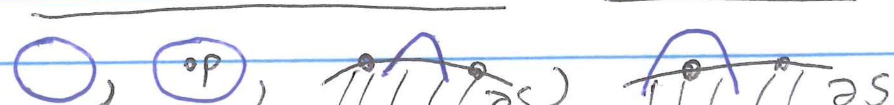


or • spiral into a puncture (e.g.  or )



OR an elementary lamination can be a closed curve (w/ no endpoints) (e.g. )

Def: An integral unbounded measured lamination on a marked surface $(S,M)$ is a finite collection of pairwise non-intersecting elem. laminations, each of which has no self-intersections & disallowing:

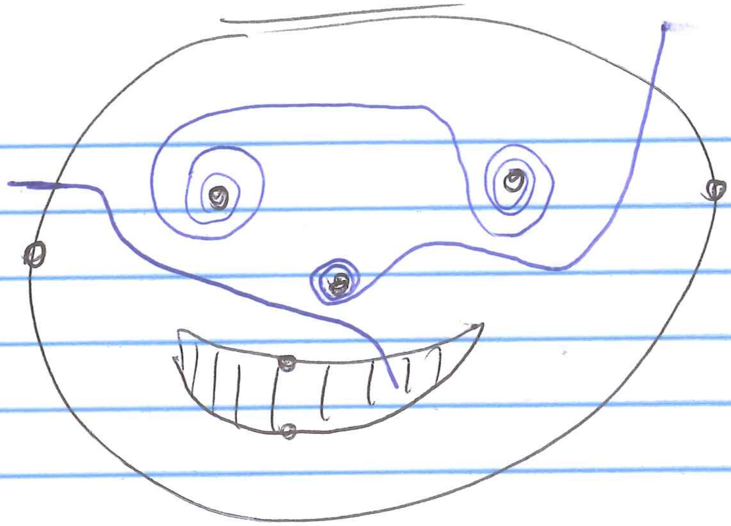


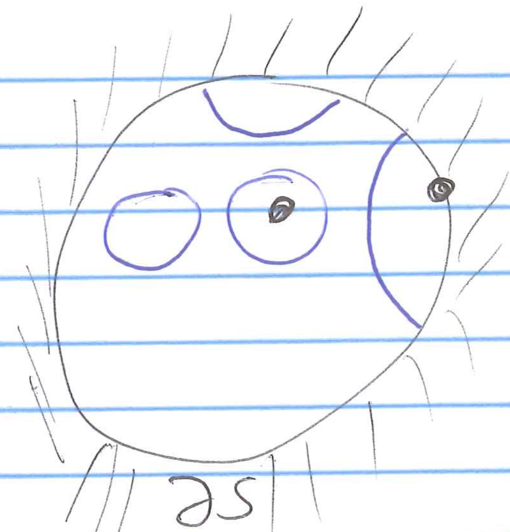


- closed curves that are contractible or around one puncture
- curves w/ two endpts in $\partial S$ isotopic to boundary containing 0 or 1 marked pt.

(5)

# Allowed | Disallowed





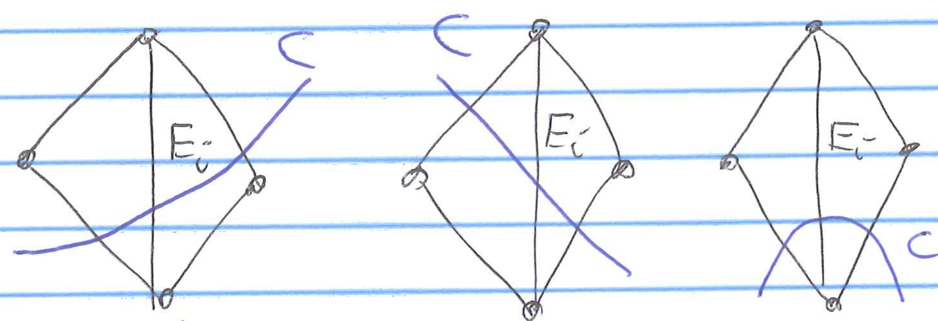

**Laminations** considered up to isotopy of curves relative to M

Given a choice of (IUM) lamination $L$ and a Triangulation $T$ of $(S,M)$,

we define $b_L(E_i; T)$, for each $E_i \in T$, by

$$b_L(E_i; T) = \sum_{\substack{\text{curves } c \text{ in } L \\ \text{crossing } E_i}} b_c^{\Diamond}(E_i) \quad \text{where}$$

$b_c^{\Diamond}(E_i) =$



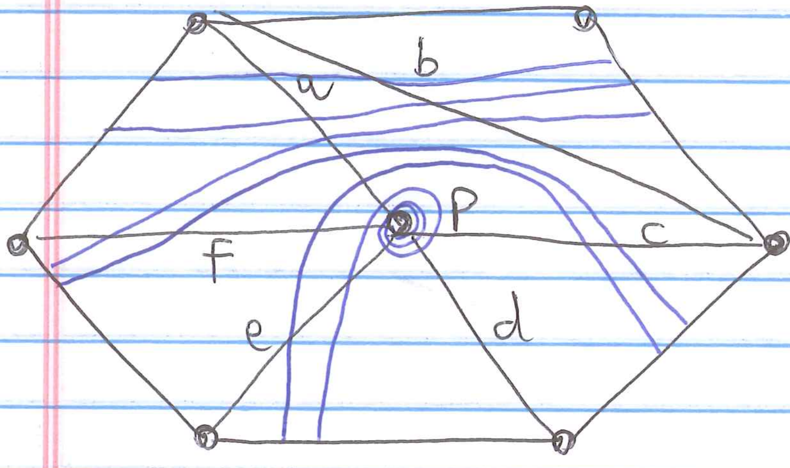

$+1$ — \$

$-1$ — Zilch

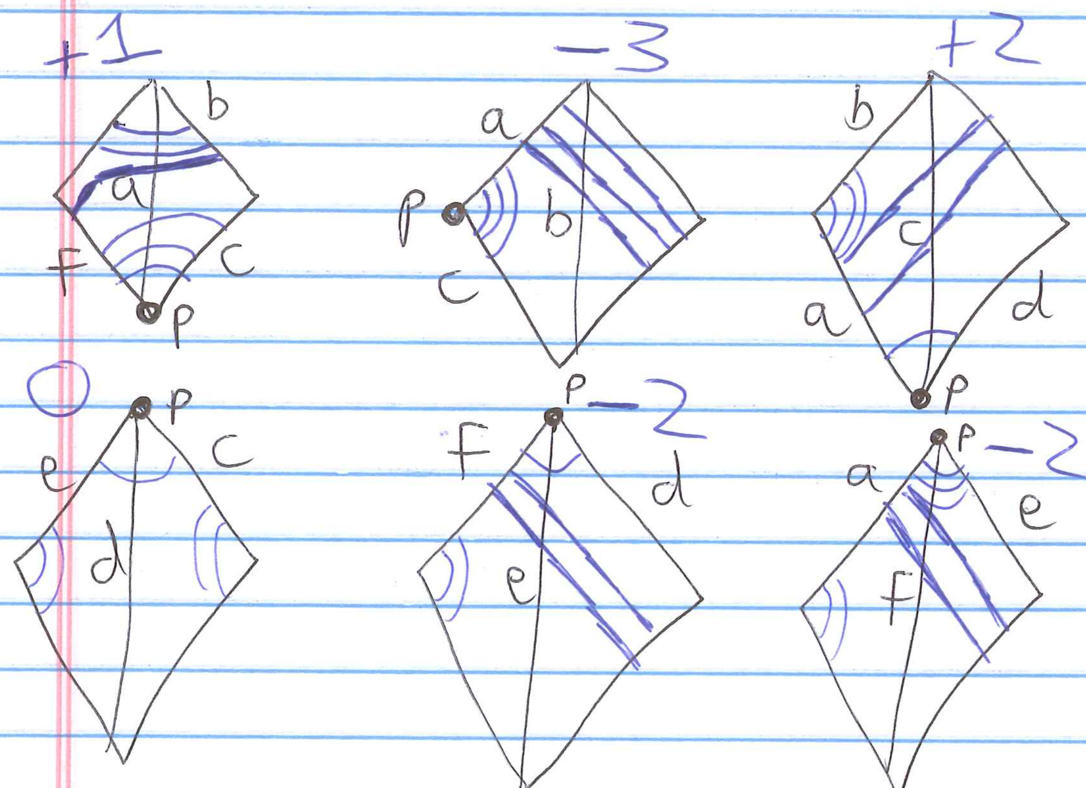
$0$ — adjacent sides

$\boxed{\text{Called Shear Coordinates}}$

Example (Fig 31 of Fomin-Thurston)




Triangulation $T$ with Lamination $L$




So $b_L(a;T)=1$, $b_L(b;T)=-3$, etc

We next flip $b \to b'$ to get triangulation $T'$ and see $\boxed{\text{how } b_L(E_i', T')\text{'s change}}$.

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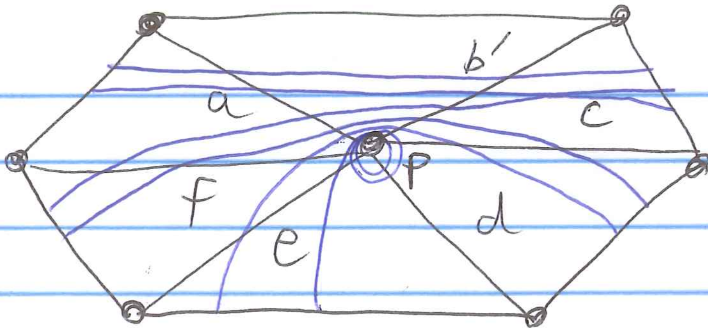
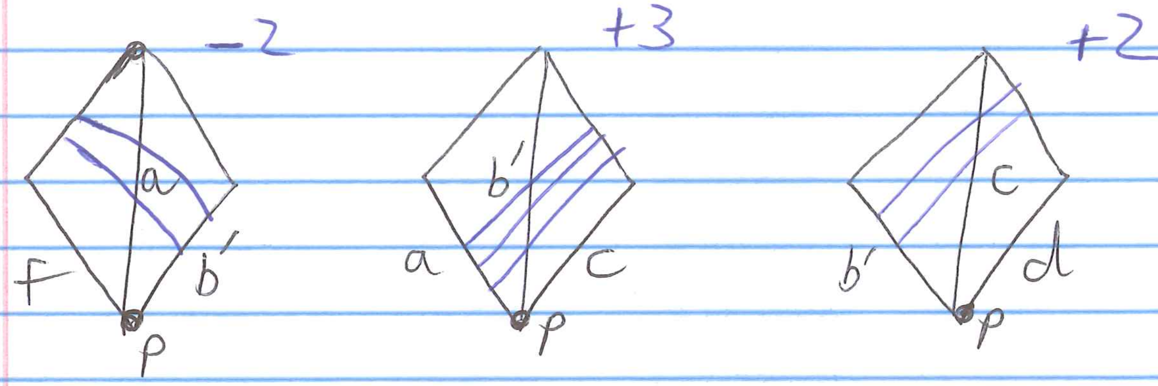
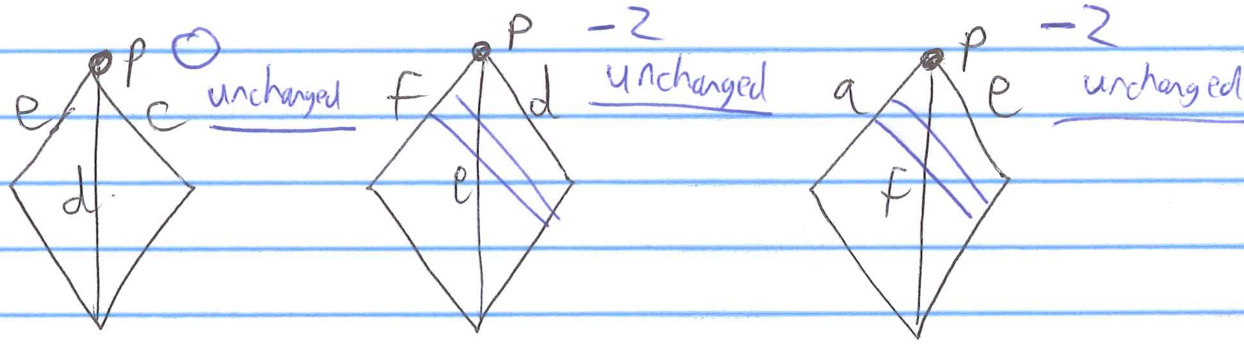
$T'$ with the same lamination $L$




$+3 = -(-3)$ $\quad b_L(b'; T') = -b_L(b; T)$

$-2 = 1 - \max(-(-3), 0)$ $\quad b_L(a; T') = b_L(a; T) - \max(-b_L(b; T), 0)$

$+2 = 2 + \max(-3, 0)$ $\quad b_L(c; T') = b_L(c; T') + \max(b_L(b; T), 0)$

$b_L(E_i; T') = b_L(E_i; T)$ o.w. (i.e. for $E_i = d, e, f$)

$\boxed{\text{This is a general phenomenom}}$

Tropicalization of the cross-ratio, i.e. $\tau$-coordinates!

Trop: $(+, \times) \longrightarrow (\max, +)$

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Thm (William Thurston) For a fixed triangulation $T$ without self-folded triangles, the map

$$L \longrightarrow \{ b_L(E_i; T) \}_{E_i \in T} \quad \text{is a } \underline{\text{bijection}}$$

$$\begin{matrix}\text{integral unbounded} \\ \text{measured laminations}\end{matrix} \longleftrightarrow \mathbb{Z}^n$$

In fact, $b_L(E_i;T)$'s transform like a row of exchange matrix under mutation.

$\Rightarrow$ Thm (W. Thurston, Fock-Goncharov) If $T$ & $T'$ are triangulations of $(S,M)$ w/o self-folded triangles related to one another by flipping $E_K$.

Then $\tilde{B}(T, \{L_j\}_{j=1}^m)$ & $\tilde{B}(T', \{L_j\}_{j=1}^m)$ related by $\mu_K$

where $\tilde{B}(T, L) := \left[\begin{array}{c} B(T) \\ \hline B(L_j)_{j=1}^m \end{array}\right] = \left[\begin{array}{c} B(T) \\ \hline B(L_1) \\ B(L_2) \\ \vdots \\ B(L_m) \end{array}\right]$

for a choice of $m$ different laminations

Each $L_j \longmapsto$ row $[b_{L_j}(E_1;T), \ldots, b_{L_j}(E_n;T)]$