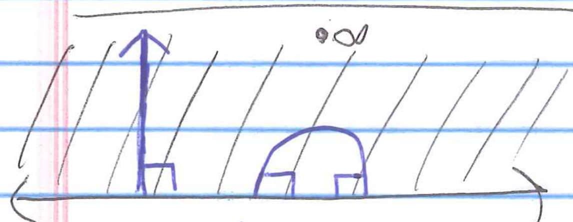
10/8/18 ① Hyperbolic Plane is the unique complete simply-connected Riemann surface w/ curvature -1. $\mathbb{H}$

Poincaré disk Model $\quad ds^2 = \dfrac{dx^2+dy^2}{(1-r^2)^2}$

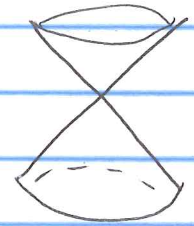
Upper Half-plane Model $\quad ds^2 = \dfrac{dx^2+dy^2}{y^2}$



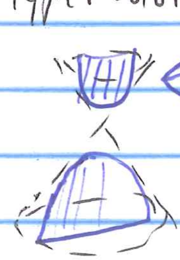
real line $\subset \mathbb{C} \cup \{\infty\}$

geodesics

Hyperboloid Model : cone $x^2+y^2-z^2=0 \subset \mathbb{R}^3$



positive Hyperboloid $= \left\{ (x,y,z) : \begin{array}{l} x^2+y^2=z^2-1 \\ z>0 \end{array} \right\}$



$dx^2+dy^2-dz^2$ on $\mathbb{R}^3$ is Lorentzian metric

Fact: Orientation-preserving isometries of upper half plane are $PSL_2(\mathbb{R}) \subset PSL_2(\mathbb{C})$.

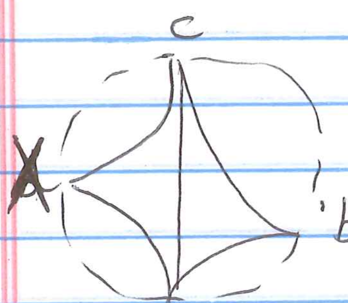
Fact: Any three distinct points in $\mathbb{R} \cup \{\infty\} = \mathbb{RP}^1$ (real projective line) can be taken to three others by a unique element of $PGL_2(\mathbb{R})$.

$PSL_2(\mathbb{R})$ index 2 in $PGL_2(\mathbb{R})$ [orientation-preserving as opposed to reversing]

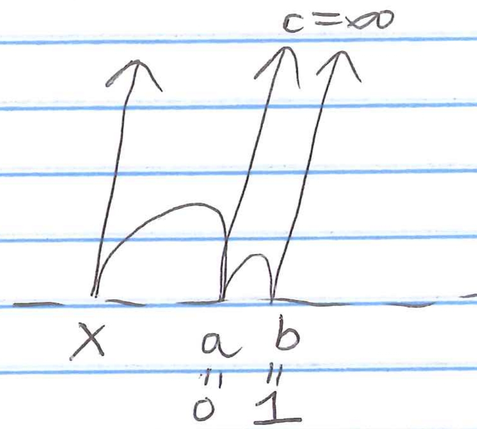
(2) Example: To send $\begin{array}{l} a \mapsto 0 \\ b \mapsto 1 \\ c \mapsto \infty \end{array}$ (for $a, b, c \in \mathbb{R}P^1$)

Consider the map $$x \mapsto \frac{(x-a)(b-c)}{(x-c)(b-a)}$$

Equivalently, we get a <u>hyperbolic quadrilateral</u>

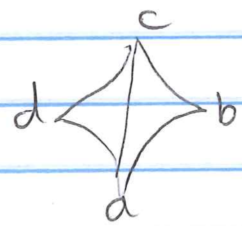


OR



<u>in Poincaré Disk</u> $\qquad$ in <u>Upper Half-Plane</u>

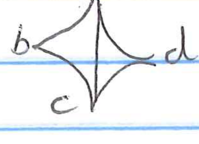
Hence moduli space of ideal triangles is zero dimensional
but " " quadrilaterals is <u>one-dimensional</u>.

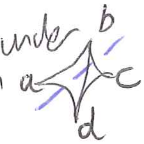
For general , determined by $\dfrac{(d-a)(b-c)}{(d-c)(b-a)} \in \mathbb{R} \cup \{\infty\}$

called the <u>cross-ratio</u>.

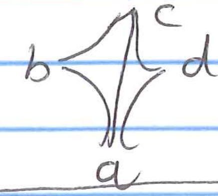
Let $\chi(d,c,b,a)$ denote the cross ratio.

- Note that χ is a <u>projective invariant</u>, i.e. $\chi(\lambda d, \lambda c, \lambda b, \lambda a) = \chi(d, c, b, a)$

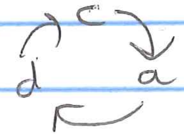
- Also $\chi(d,c,b,a) = \chi(b,a,d,c)$  $\boxed{\dfrac{(b-c)(d-a)}{(b-a)(d-c)}}$
so preserved under 180° rotation.

∘ Similarly invariant under diagonal reflection 

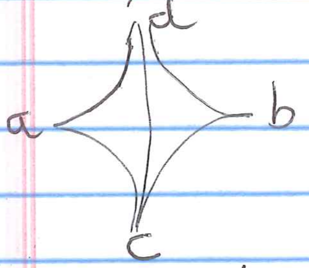
$x(a,b,c,d) = \frac{(a-d)(c-b)}{(a-b)(c-d)} = x(d,c,b,a)$

3 • On the other hand, vertical or horizontal reflection sends $\boxed{x \longmapsto x^{-1}}$, e.g. 

has $x^{-1} = \frac{(b-a)(d-c)}{(b-c)(d-a)}$

• 3-cycles like 

sends $\boxed{x \longmapsto 1 - x^{-1}}$

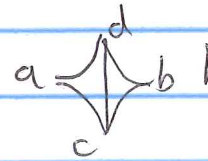


has cross-ratio $\frac{(a-c)(b-d)}{(a-d)(b-c)}$

Note: $1 - x = \frac{(d-c)(b-a)}{(d-c)(b-a)} - \frac{(d-a)(b-c)}{(d-c)(b-a)}$

$-bc - ad \qquad +ab + cd$

$$\frac{(\cancel{bd} - bc + \cancel{ac} - ad) - (\cancel{bd} - ab - cd + \cancel{ac})}{(d-c)(b-a)} = \frac{(a-c)(b-d)}{(d-c)(b-a)}$$

$\Rightarrow$ 

has cross-ratio $-x^{-1} \cdot (1-x) = 1 - x^{-1}$.

negative since multiply through by $\frac{(d-c)(b-a)}{(a-d)(b-c)}$

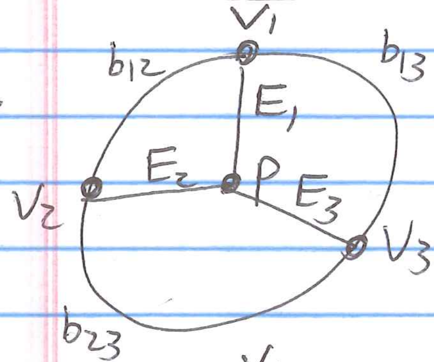
Note: $x \mapsto \underset{= \frac{x-1}{x}}{1 - \frac{1}{x}} \mapsto 1 - \frac{x}{x-1} = \frac{1}{1-x} \mapsto x$ (indeed order 3)

$\langle x \mapsto 1 - x^{-1},\ x \mapsto x^{-1} \rangle$ generates order 6 subgp $\cong S_3$ inside of S_4 (permuting a,b,c,d)

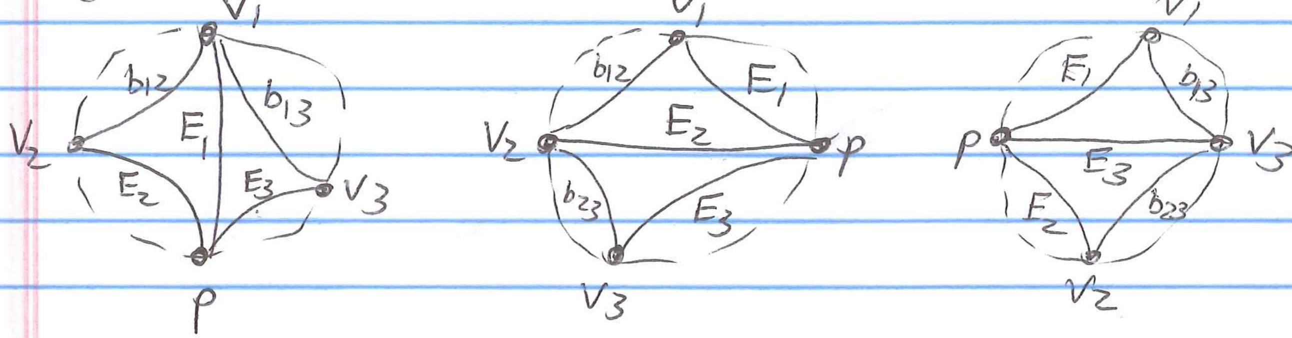
The above definition of shear coordinates depended on the local embedding of an ideal quadrilateral into $\mathbb{H}$. We can do this for each inscribed quadrilateral (for each $E_i \in T$).

④ We apply these local embeddings ~~for~~ for inscribed quadrilaterals even for punctured polygons.

E.g.

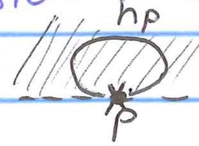


$\longmapsto \mathbb{H}$ via



Here is a different way to view the shear coords without the embedding:

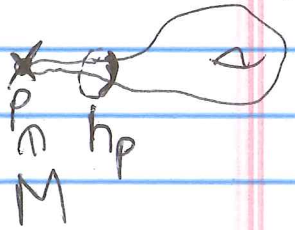
Def: A horocycle h_p at an ideal point p (i.e. at circle at ∞ in Poincaré Disk Model or on y-axis in Upper Half-plane) is ~~a set of points all equidistant to p.~~ a circle orthogonal to any geodesic based at p.

Topologically:  h_p | Geometrically:  or 

Def: Decorated Teichmüller Space $\widetilde{\mathcal{T}}(S,M)$

parametrized by

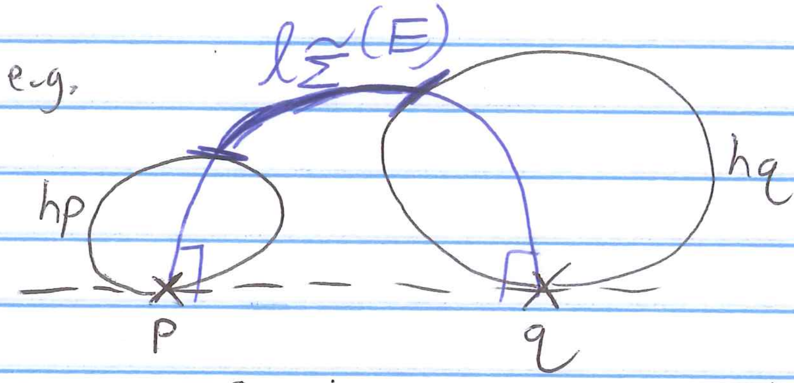
- a point in $\mathcal{T}(S,M)$
- a choice of horocycle at each cusp of M.



(5) **Def (Penner):** For an arc E on (S, M) and a choice $\widetilde{\Sigma} \in \widetilde{\mathcal{T}}(S,M)$, the length $\ell_{\widetilde{\Sigma}}(E)$ defined as

$$\ell_{\widetilde{\Sigma}}(E) := \text{length of the geodesic representative of } E \text{ between horocycles}$$

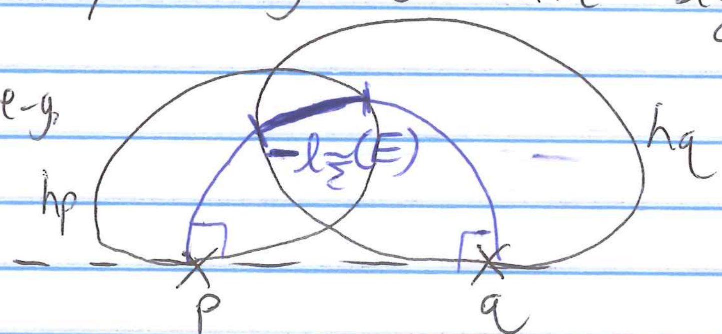
e.g.



E from p to q

Rem: If horocycles chosen large enough so that they intersect, we define $\ell_{\widetilde{\Sigma}}(E)$ to be negative:

e.g.



Def: The lambda length (λ-length) of E defined as

$$\lambda_{\widetilde{\Sigma}}(E) := e^{\frac{\ell_{\widetilde{\Sigma}}(E)}{2}} \in \mathbb{R}_{>0} \quad \left(\text{usual exponential map}\right)$$

Called Penner coordinates

Thm (Penner): For any triangulation $T = \{E_i\}_{i=1}^{n}$ w/o self-folded triangles, the map

$$\prod_{\gamma \in T \cup \{\text{Boundary Arcs}\}} \lambda(\gamma) : \widetilde{\mathcal{T}}(S,M) \to \mathbb{R}_{>0}^{n+c}$$

is a homeomorphism.