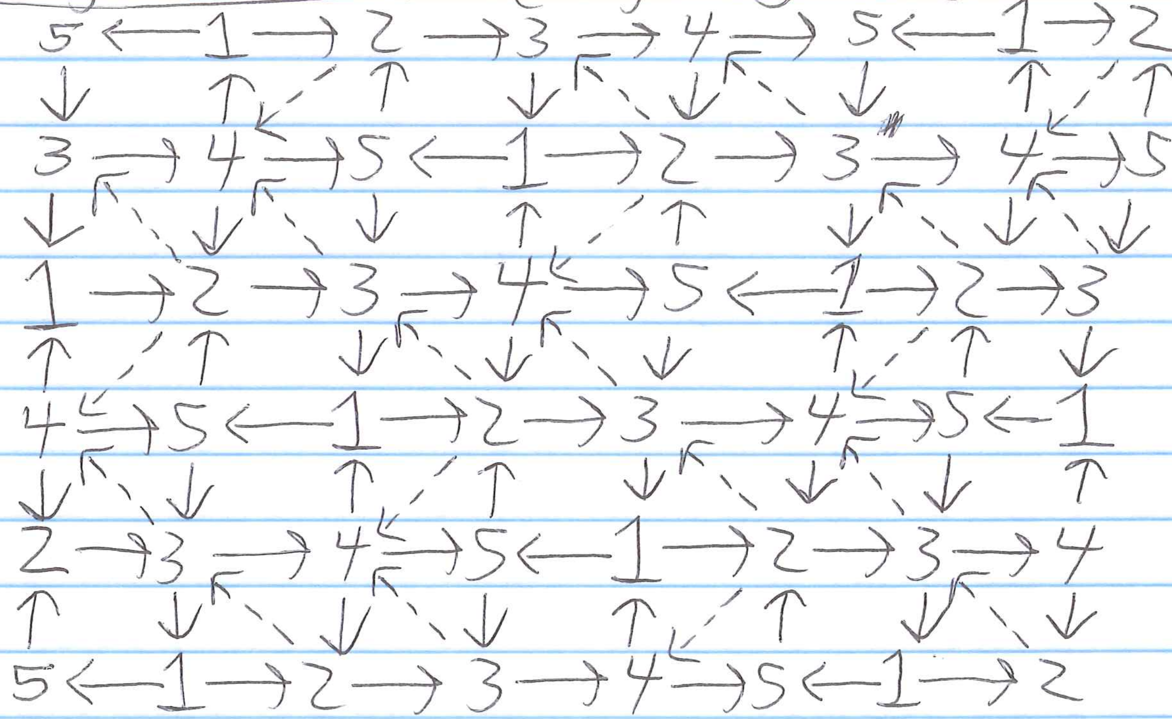
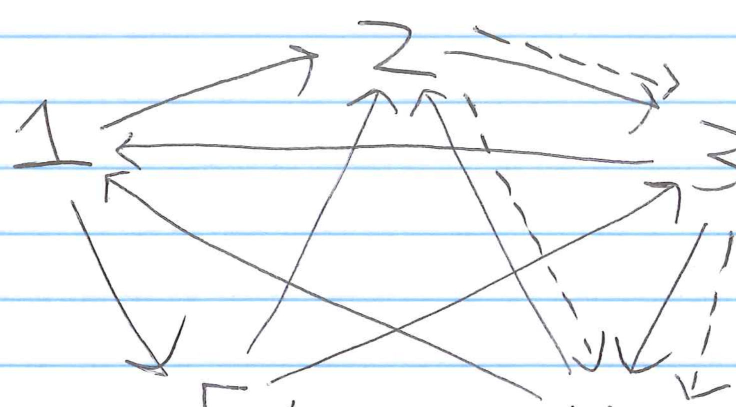
11/14/18

Combinatorial Interpretations for F-polynomials and Cluster Variables w/ Principal Coefficients associated to Gale-Robinson sequences

Recall our construction of a (r,s,N)-Gale-Robinson quiver on a torus from last class.

E.g. Somos-5 $(r=1,\ s=2,\ N=5)$



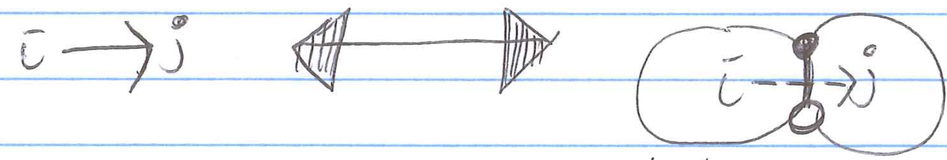


1-periodic quiver giving rise to

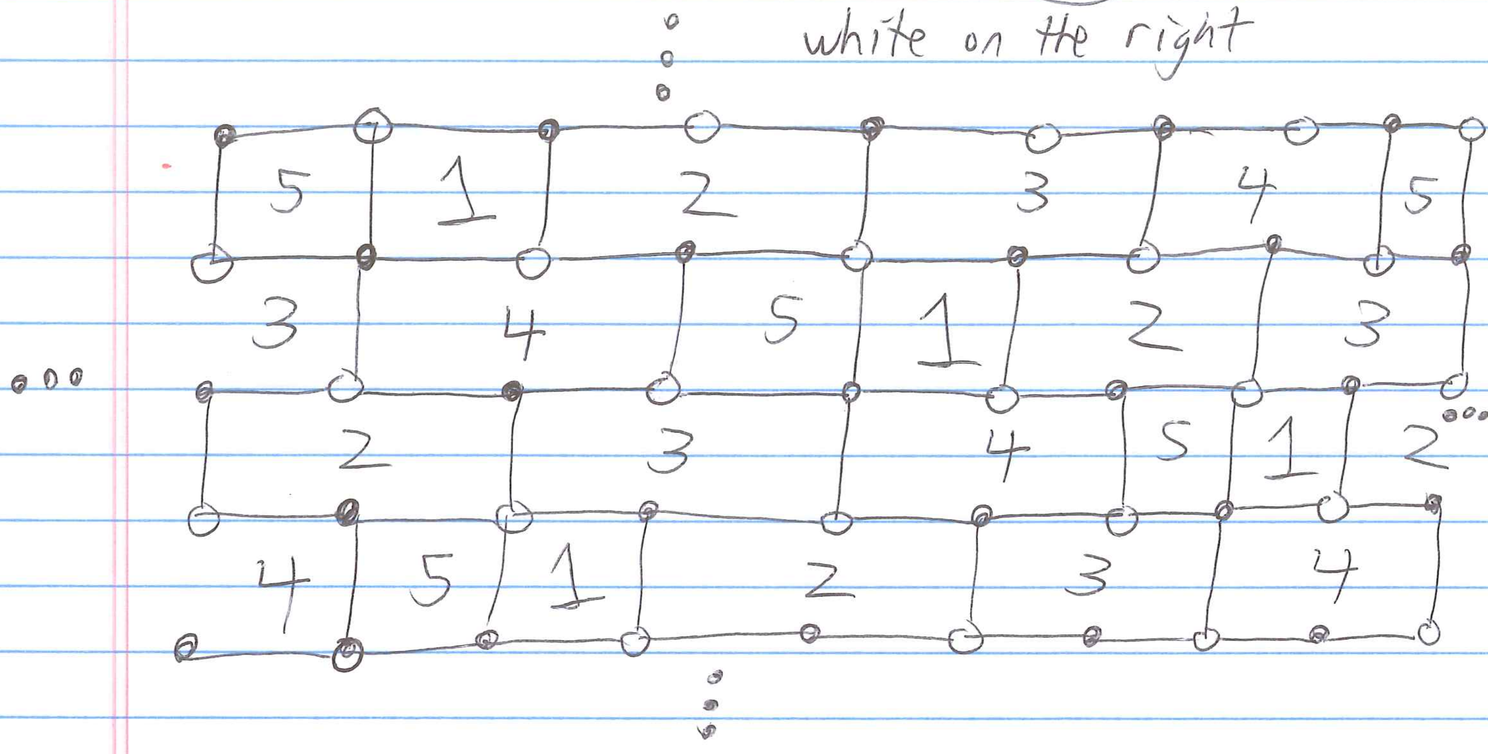
$$\boxed{x_n x_{n-5} = x_{n-1}x_{n-4} + x_{n-2}x_{n-3}}$$

if we mutate $1,2,3,4,5,1,2\ldots$

(2) We can dualize to a tiling on a torus by

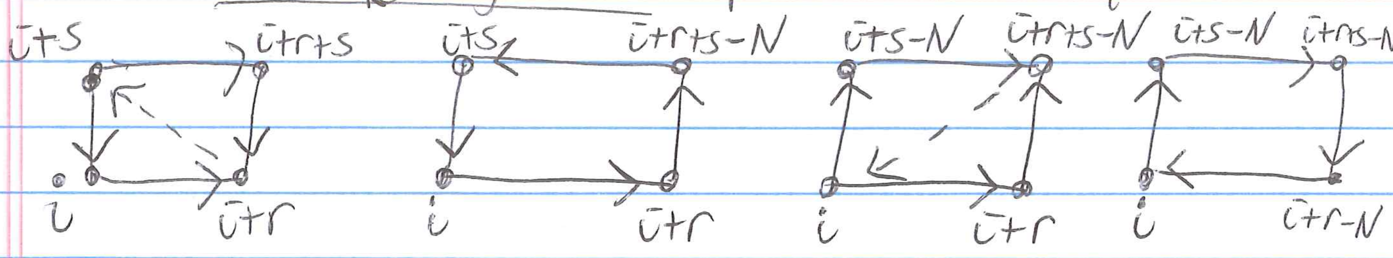


white on the right

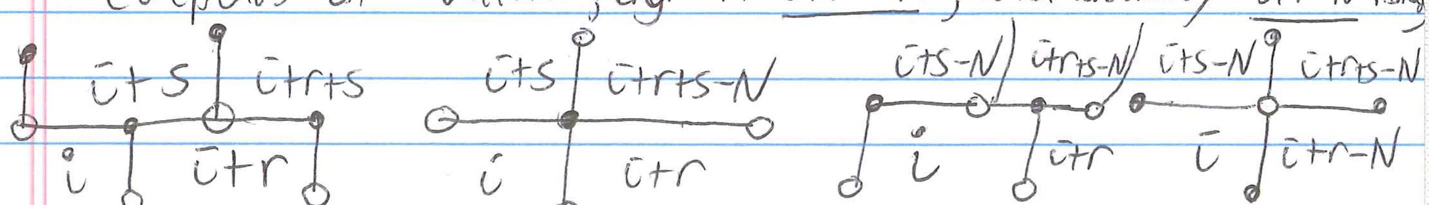


This construction would similarly work for other Gale-Robinson quivers on the torus.

Four local configurations possible in the quiver:



(depends on values, e.g. if $i+r>N$, vertex labeled by $i+r-N$ instead)



③ **Claim 1**: If we build corresponding quiver w/ principal coefficients via the convention $\boxed{i'} \leftarrow i$ for $1 \le i \le N$, and mutate $1, 2, 3, \ldots, N, 1, 2, \ldots,$ then the Gale-Robinson recurrence becomes

$$\boxed{x_n x_{n-N} = x_{n-r} x_{n-N+r} + \prod_{i=1}^{N} y_i^{d(n-N-i,s,N-s)} x_{n-s} x_{n-N+s}}$$

Here $d(n-N-i, s, N-s) = \#\{(A,B) \in \mathbb{Z}^2_{\ge 0}$ such that

$$(n-N-i) = A \cdot s + B \cdot (N-s)\}.$$

$d(m,a,b)$ is known as the restricted partition function and due to a Theorem of Popoviciu,

$$d(m,a,b) = \frac{m}{ab} - \left\{\frac{b^{-1}n}{a}\right\} - \left\{\frac{a^{-1}n}{b}\right\} + 1 \quad \text{where}$$

$a^{-1} :=$ inverse of a modulo b, $\quad \left\{\frac{c}{d}\right\} :=$ fractional part of c/d.
$b^{-1} :=$ inverse of b modulo a,

Eg (Somos-5): $x_6 x_1 = x_5 x_2 + \underline{y_1}\, x_3 x_4$

use # pos integer combos of 2 & 3.
$2 = s$, $3 = N-s$

$$x_7 x_2 = x_6 x_3 + \underline{y_2}\, x_4 x_5$$

$$x_8 x_3 = x_7 x_4 + \underline{y_1 y_3}\, x_5 x_6$$

$$x_9 x_4 = x_8 x_5 + \underline{y_1 y_2 y_4}\, x_6 x_7$$

$$x_{10} x_5 = x_9 x_6 + \underline{y_1 y_2 y_3 y_5}\, x_7 x_8$$

4

$$X_{11}X_6 = X_{10}X_7 + \underline{y_1 y_2 y_3 y_4}\; X_8 X_9$$

$y_6 \equiv y_1$ ④ ③ ②

$$X_{12}X_7 = X_{11}X_8 + \underline{y_1^2 y_2 y_3 y_4 y_5}\; X_9 X_{10}$$

⋮

$y_2 \equiv y_7$ $\qquad y_1 y_2 y_3 y_4 y_5 y_7$ ⑥ ⑤ ④ ③ ②

⑥ $= 2+2+2 = 3+3$ hence y_1^2.

PF: Unlike Octahedron Recurrence quiver or $A_n \times A_m$ quivers mutations along $1, 2, 3, \ldots, N, 1, 2, \ldots$ don't commute.

But since built to be 1-periodic, everytime mutating at vertex i in this sequence, looks locally like

$$\begin{tikzcd} & i+s \arrow[d] & \\ i-r & i \arrow[l] \arrow[r] & i+r \\ & i-s \arrow[u] & \end{tikzcd}$$

Thus w/ principal coeffs $\boxed{i'} \leftarrow i$

arrows spread out to

$$\begin{tikzcd} i+s \arrow[d] \\ \boxed{i'} \\ i-s \arrow[u] \end{tikzcd}$$

from mutation (rotation/reverse of our earlier convention)

Hence y-term is coefficient of $X_{n-s}X_{n+N+s}$ term

and of form $\displaystyle\prod_{k=-m}^{m} y_{i+ks \bmod N}$ after m mutations

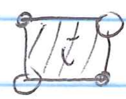
Algebraic manipulations yield the restricted partition function as above.

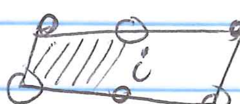
(5) Theorem: Given cluster variable x_n (with principal coefficients) defined as above, x_n has a Laurent expansion

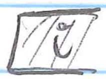
$$x_n = \sum_{\substack{M \text{ perfect} \\ \text{matching of } G_n^{(r,s,N)}}} x(M)\, y(M) \quad \text{where}$$

$G_n^{(r,s,N)}$ constructed as follows

Step 1: If we write $n = (rK + i) + N$ (i.e. working mod r), where $1 \le i \le r$, locate a face labeled i in the tiling dual to the (r,s,N)-Gale-Robinson quiver.



If it is a rectangle  select the left-hand square of this face. (If it is square, select it)

Step 2: Build an Aztec Diamond of size $K+1$ (i.e. height & width $2K+1$) centered so that the selected square  is the leftmost square of the Aztec Diamond.

Step 3: After superimposing, will have leaf edges. Force perfect matchings until every vertex has valence at least 2.

$y(M)$ defined as $\prod_{i=1}^{N} y_i^{h_i}$ where h_i = # Faces labeled i that were twisted from minimal matching

all horizontal

⑥ $x(M)$ given by face weighting

If face F_i is contained in $G_n^{r,s,N}$ (and labelled i)

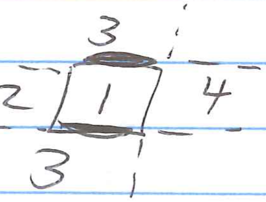
let $\varepsilon(F_i) = \left[s-1-\#(M\cap\partial F_i) \right]$ if F_i is a $2s$-gon.

If face F_i borders $G_n^{r,s,N}$ "an open face"

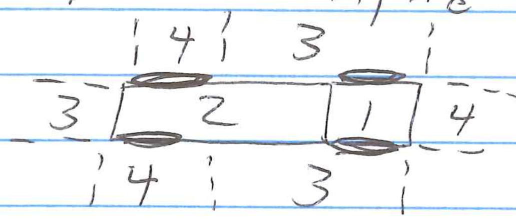
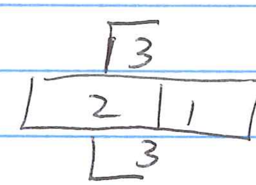
$$\varepsilon(F_i) = \left\lceil \frac{s}{2} \right\rceil - \#(M\cap\partial F_i)$$

if the border of F_i & $G_n^{r,s,N}$ has s edges.

Then $x(M) = \prod_{\substack{F_i \text{ contained in} \\ \text{or bordering } G_n^{r,s,N}}} x_i^{\varepsilon(F_i)}$ [Laurent monomial]

E.g. Somos 4: $G_5^{1,3,4} =$  $x_5 = \frac{x_2x_4}{x_1} + \frac{y_1 x_3^2}{x_1}$

$$x_6 = \frac{y_1y_2\,\cancel{x_2}\,x_4^{2}}{\cancel{x_1}\,x_2} + \frac{\cancel{x_2}\,x_3x_4}{x_1\cancel{x_2}} + \frac{y_1 x_3^3}{x_1x_2}$$

$G_6^{1,3,4} =$  from 

Proof of Thm next time by Speyer's general method for Octahedron Recurrence & its projections.