\[ \Gamma(S,M) = \{ \text{pt in } \Gamma(S,M) + \text{choice of } \text{horocycle at every cusp in } M \} \]

Horocycle based at \( m \in M \) is a circle orthogonal to any geodesic passing through \( m \) (embedded onto \( \mathbb{RP}^1 \) in \( \mathbb{H} \)).

\[ \lambda(E) = \ell(E) \]

\( \lambda \)-length of arc \( E \) in \( \Gamma(S,M) \) is \( \frac{e^2}{\ell(E)} \) where \( \ell(E) \) is geodesic length between horocycles about two endpoints of \( E \).

Claim: For an ideal quadrilateral inscribing arc \( E \) in Ptolemy Disk in Upper Half-plane

\[ \chi = \text{cross-ratio} = \frac{(d-a)(b-c)}{(d-c)(b-a)} = \frac{-\lambda(A)\lambda(C)}{\lambda(D)\lambda(B)} \]
Thm (Penner): For any triangulation $T = \{ E_i \}_{i=1}^n$ w/o self-touled triangles, the map $\prod_{T \in \mathcal{T}(S_M)} \lambda \Sigma(\sigma) : \mathcal{F}(S_M) \to \mathbb{R}^{n_{tr}}$

(defined by $\Sigma \in \mathcal{F}(S_M)$) is a homeomorphism.

Further, the ratio $\frac{\lambda \Sigma(A) \lambda \Sigma(C)}{\lambda \Sigma(D) \lambda \Sigma(B)}$ from an ideal quadrilateral is independent of the choices of horocycles and we recover homeomorphism $\prod_{T \in \mathcal{T}(S_M)} \lambda \Sigma(\sigma) : \mathcal{F}(S_M) \to \mathbb{R}^n$.

Let us develop some hyperbolic geometry to better explain the relationship between cross-ratio coords, $\lambda$-lengths, and arc lengths of horocycles. (Following D. Thurston 2012 notes)

Consider the Upper Half-Plane Model w/ $ds = \sqrt{dx^2 + dy^2}$

Claim: Hyperbolic length $\lambda$ yields $\lambda$-length $\frac{d}{\lambda} = \frac{x}{\sqrt{\Delta_1 \Delta_2}}$

where $x = \text{Euclidean distance between ideal points } p_1, p_2 \in \mathbb{R}^2$

$\Delta_i = \text{Euclidean diameter of horocycle } h_i \text{ based at } p_i$.

Claim: If $p_2 = \infty$, $\lambda$-length $= \frac{\sqrt{y_2}}{\sqrt{\Delta_1}}$. 
Firstly, we show that $l(A), l(B), l(c)$ can each be freely and independently chosen from $\mathbb{R}$. Starting w/ arbitrary horocycles on an ideal triangle

We decrease the radii of the horocycles by $\alpha, \beta$, and $\gamma$, respectively (with possible negative values for increasing).

The new hyperbolic lengths are

$$
\begin{align*}
l(A) &= l_0(A) + \beta + \gamma + \alpha \quad \text{(can freely choose } \alpha, \beta, \gamma \in \mathbb{R} \text{ to obtain)} \\
l(B) &= l_0(B) + \gamma + \alpha + \beta \\
l(c) &= l_0(c) + \alpha + \beta
\end{align*}
$$

Special case: $l(A) = l(B) = l(c) = 0$.

A specific choice of horocycles so all pairwise tangent to each other and the ideal boundary.

Rem: Up to $\text{PSL}_2(\mathbb{R})$, i.e., Möbius transformations,

Here is a unique such configuration of 4 pairwise tangent circles.

We now compute hyperbolic length $l$ in case $P_2 = \infty$

$$
l = \int_{\Delta_1}^{y_2} \frac{dy}{y} = \log(y_2) - \log(\Delta_1)
$$

since no $x$-movement along geodesic in this case

$$
\Rightarrow \quad \lambda = e^{l/2} = \sqrt{\frac{y_2}{\Delta_1}} \quad \text{as desired}.
$$
Notice that as we scale $h_2$, the horocycle at $\alpha$, 
$h_2 \odot \gamma = \gamma_2 \rightarrow h_2' \odot \gamma = \alpha \gamma_2$, that changes 
the $\lambda$-length by $\sqrt{\alpha}$. [if $\alpha > 1$, $h_2'$ closer to $\infty$] 
so $h_2'$ "shrunk" compared to $h_2$.

Similarly rescaling $h_1$ (by dividing its diameter by $\alpha$) 
also rescales by $\sqrt{\alpha}$.

We now consider the case $p_1, p_2 \in \mathbb{R}$ (neither $\infty$)

* Observe that if $h_1$ & $h_2$ are tangent, $l = 0 \implies \lambda = 1$

\[ \text{e.g. if } x = 1, \triangle_1 = 1, \triangle_2 = 1 \] diameters

* From above, if we rescale either of these horocycles by $\lambda$-

$\lambda$-length also rescales by $\sqrt{\alpha}$,

\[ \text{so } x = 1, \triangle_1 = \frac{1}{\alpha}, \triangle_2 = 1 \implies \lambda = \sqrt{\alpha} \]

\[ \text{while } \triangle_2 = \frac{1}{\beta} \implies \lambda = \sqrt{\alpha \beta} \]

Also, we can dilate the entire picture & $l$ scales with

so if $x \in \mathbb{R}$, $\triangle_1 = \frac{x}{\alpha}, \triangle_2 = \frac{x}{\beta} \implies \lambda = \sqrt{\alpha \beta}$

We conclude that \[ \lambda = \frac{x}{\sqrt{\triangle_1 \triangle_2}} \]

\[ \left( \sqrt{\frac{x}{\triangle_1}}, \sqrt{\frac{x}{\triangle_2}} \right) \]

as desired.

\[ (\text{without needing to use integral w/) } ds = \sqrt{dx^2 + dy^2} \]

\[ \frac{1}{y} \text{ over } \]
Cor: \( X(E) = \lambda(A) \lambda(C) \) for \( X(D) \lambda(B) \)

PF: \( \lambda(A) = -\chi \frac{\Delta b}{\Delta a} \), \( \lambda(B) = \frac{1}{\sqrt{\Delta a \Delta b}} \).

\( \lambda(C) = \frac{\chi_c}{\sqrt{\Delta b}} \), \( \lambda(D) = \frac{\chi_c}{\sqrt{\Delta d}} \).

Note the sign

Cor: For a decorated ideal triangle, let \( L(h_c) \) be hyperbolic length of an arc segment along \( h_c \) between \( A \& B \).

Then \( L(h_c) = \frac{\lambda(C)}{\lambda(A) \lambda(B)} \).

PF: Consider Upper-Half-Plane Model.

\[ L(h_c) = \int_{x=0}^{s} \frac{dx}{\chi_c} = \int_{0}^{s} \frac{x}{\chi_c} \text{d}x = \frac{s}{\chi_c} \]

and from above \( \lambda(C) = \frac{\chi_c}{\sqrt{\Delta a \Delta b}} \sqrt{\frac{\chi_c}{\chi_c \Delta b}} \).

Cor: (Ptolemy Lemma for Hyperbolic Quadrilateral)

\( \lambda(E) \lambda(F) = \lambda(A) \lambda(C) + \lambda(B) \lambda(D) \).
6. \( \text{PF: } \frac{D}{C} \sim \frac{E}{A} \)

\[
L(h_F) = L(h_a) + L(h_b)
\]

\[
\lambda(F) - \frac{\lambda(A)}{\lambda(C)} + \frac{\lambda(B)}{\lambda(E)}
\]

Can also encode choice of ideal pt in \( \text{RP}^1 \) plus a horocycle as \( (x,y) \in \mathbb{R}^2 \mapsto \frac{x}{y}\).

Claim: Under this correspondence, \( \lambda_E = \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \).

\[
\lambda_E = \frac{\frac{x_2}{y_2} - \frac{x_1}{y_1}}{\sqrt{\frac{1}{y_1^2}} \sqrt{\frac{1}{y_2^2}}}
\]

Yields: Direct translation between Ptolemy Relation and Plücker Relation for

Cor: For Markoff Cl. Alg, \[
\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}
\]

\[
MI = \frac{x^2 + y^2 + z^2}{xyz}
\]

Identify \( h_{hp} \) length w/ associated edge.

\[
L(h_{hp}) = L(h_{x_1}) + L(h_{y_2}) + L(h_{z_1}) + L(h_{x_2}) + L(h_{y_1}) + L(z_2) = 2 \cdot MI.
\]