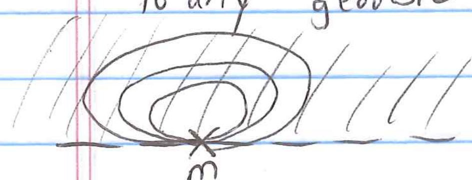
① 10/10/18 Review: $\mathcal{T}(S,M) = \{\text{certain hyperbolic metrics}\}/\sim$ Diffeo isotopic to id.

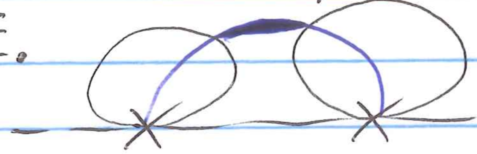
$\tilde{\mathcal{T}}(S,M) = \left\{ \begin{array}{l} \text{pt in } \mathcal{T}(S,M) + \text{choice of} \\ \underline{\text{horocycle}} \text{ at every cusp in } M \end{array} \right\}$

Horocycle based at $m \in M$ is a circle orthogonal to any geodesic passing through m (embedded onto $\mathbb{RP}^1$ in $\overline{\mathbb{H}}$)

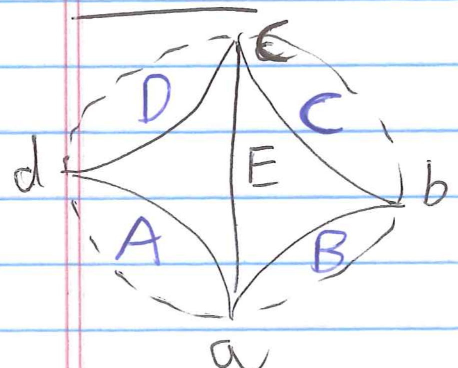


[Equivalently, the limit set of circles through a fixed point as center moves to the real line boundary]
(see bottom left)

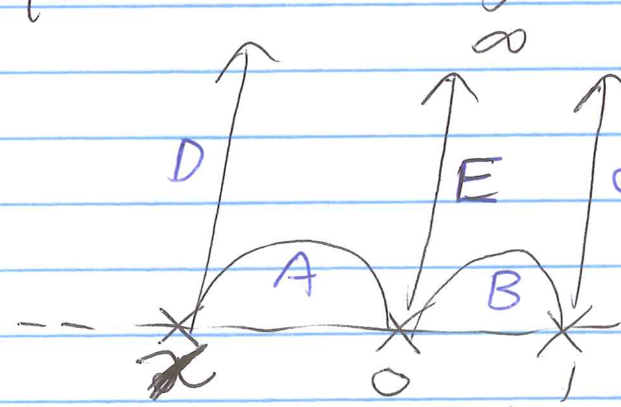
λ-length of arc E in (S,M) is $\lambda(E) = e^{\frac{\ell(E)}{2}}$ where $\ell(E)$ is geodesic length between horocycles about two endpts of E.



Claim: For an ideal quadrilateral inscribing arc E



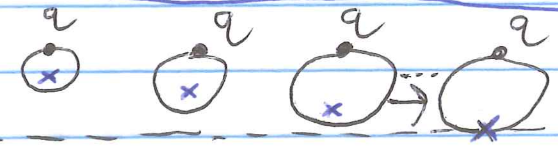
in Ptolemy Disk

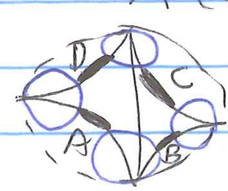


in Upper Half-plane

x = cross-ratio coordinate $:= \frac{(d-a)(b-c)}{(d-c)(b-a)}$ also $= \frac{-\lambda(A)\lambda(C)}{\lambda(D)\lambda(B)}$

Horocycle as limit circle of circles through q



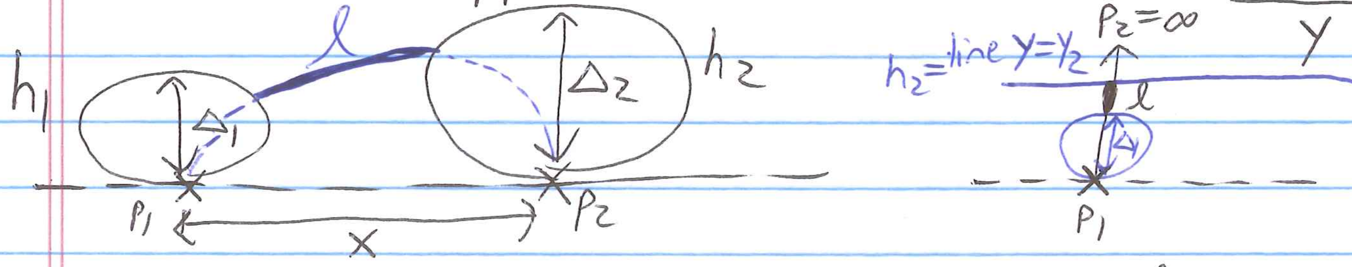


② **Thm (Penner):** For any triangulation $T = \{E_i\}_{i=1}^{n}$ w/o self-folded triangles, the map $\prod_{\gamma \in T \cup \{\text{Boundary Arcs}\}} \lambda_{\widetilde{\Sigma}}(\gamma) : \widetilde{\mathcal{T}}(S,M) \to \mathbb{R}_{>0}^{n+c}$ (defined by $\widetilde{\Sigma} \in \widetilde{\mathcal{T}}(S,M)$) is a homeomorphism.

Further, the ratio $\dfrac{\lambda_{\widetilde{\Sigma}}(A)\,\lambda_{\widetilde{\Sigma}}(C)}{\lambda_{\widetilde{\Sigma}}(D)\,\lambda_{\widetilde{\Sigma}}(B)}$ from an ideal quadrilateral is independent of the choices of horocycles, and we recover the homeomorphism $\prod_{\gamma \in T} x_{\Sigma}(\gamma) : \mathcal{T}(S,M) \to \mathbb{R}^{n}$. (by choosing $\Sigma \in \mathcal{T}(S,M)$)

Let us develop some hyperbolic geometry to better explain the relationship between cross-ratio coords, λ-lengths, and arc lengths of horocycles. (Following D. Thurston 2012 notes)

Consider the Upper Half-Plane Model w/ $ds = \dfrac{\sqrt{dx^2+dy^2}}{y}$



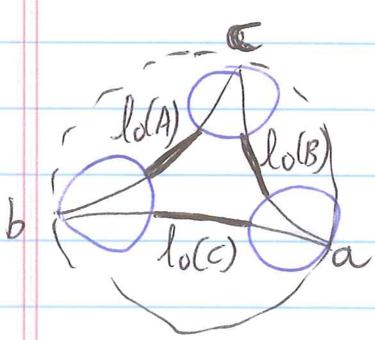
Claim: Hyperbolic length ℓ yields λ-length $e^{\frac{\ell}{2}} = \dfrac{x}{\sqrt{\Delta_1 \Delta_2}}$

where x = Euclidean distance between ideal points $p_1, p_2 \in \mathbb{R}$

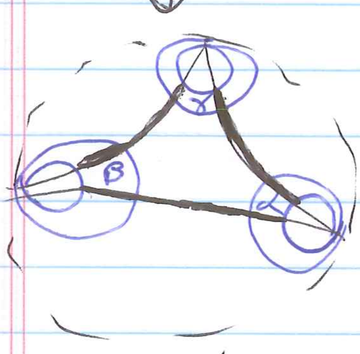
Δ_i = Euclidean diameter of horocycle h_i based at p_i.

Claim: If $p_2 = \infty$, λ-length $= \sqrt{\dfrac{y_2}{\Delta_1}}$

③ Firstly, we show that $\ell(A), \ell(B), \ell(C)$ can each be freely and independently chosen from $\mathbb{R}$. Starting w/ arbitrary horocycles on an ideal triangle



We decrease the radii of the horocycles by α, β, and γ, respectively (w/ possible negative values for increasing)



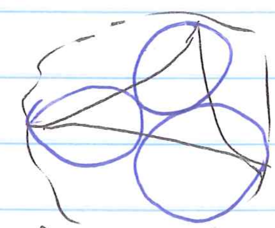
The new hyperbolic lengths are

$$\begin{aligned} \ell(A) &= l_0(A) + \beta + \gamma \\ \ell(B) &= l_0(B) + \gamma + \alpha \\ \ell(C) &= l_0(C) + \alpha + \beta \end{aligned}$$

[Can freely choose $\alpha, \beta, \gamma \in \mathbb{R}$ to obtain any values on LHS]

<u>Special case:</u> $\ell(A) = \ell(B) = \ell(C) = 0$.

A specific choice of horocycles so all pairwise tangent to each other and the ideal boundary.



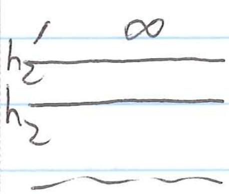
<u>Rem:</u> Up to $PSL_2(\mathbb{R})$, i.e. Möbius transformations, there is a <u>unique</u> such configuration of <u>4 pairwise tangent circles</u>.

We now compute hyperbolic length ℓ in case $P_2 = \infty$

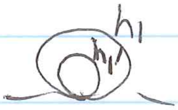
$$\boxed{\ell = \int_{y=\Delta_1}^{Y_2} \frac{dy}{y} = \log(Y_2) - \log(\Delta_1)}$$

since no x-movement along geodesic in this case

$$\Rightarrow \boxed{\lambda = e^{\ell/2} = \sqrt{\frac{Y_2}{\Delta_1}}}$$ as desired.



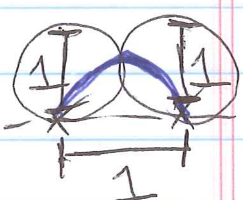
Notice that as we scale h_2, the horocycle at ∞, $h_2: y = y_2 \longrightarrow h_2': y = \alpha y_2$, that changes the λ-length by $\sqrt{\alpha}$. [if $\alpha > 1$, h_2' closer to ∞ so h_2' "shrunk" compared to h_2]



Similarly rescaling h_1 (by dividing its diameter by α) also rescales by $\sqrt{\alpha}$.

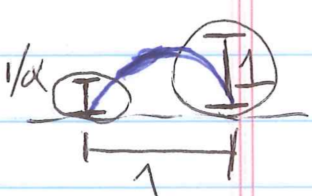
We now consider the case $p_1, p_2 \in \mathbb{R}$ (neither ∞)

- Observe that if h_1 & h_2 are tangent, $\underline{l = 0 \Rightarrow \lambda = 1}$



e.g. if $X = 1$, $\underbrace{\Delta_1 = 1, \Delta_2 = 1}_{\text{diameters}}$

- From above, if we rescale either of these horocycles by α λ-length also rescales by $\sqrt{\alpha}$,



so $X = 1, \Delta_1 = \frac{1}{\alpha}, \Delta_2 = 1 \Rightarrow \lambda = \sqrt{\alpha}$
   $\;\;\;\; {}'' \qquad\quad {}'' \quad \Delta_2 = \frac{1}{\beta} \Rightarrow \lambda = \sqrt{\alpha}\sqrt{\beta}$

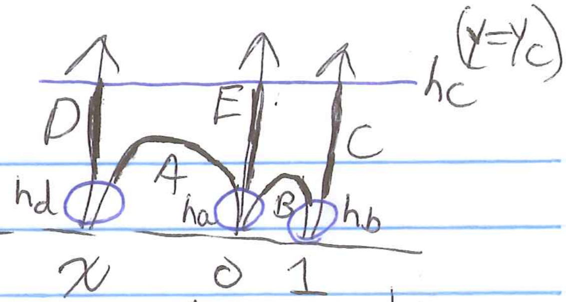
Also we can dilate the entire picture & l scales w/ X so if $X \in \mathbb{R}$, $\Delta_1 = \frac{X}{\alpha}$, $\Delta_2 = \frac{X}{\beta} \Rightarrow \lambda = \sqrt{\alpha}\sqrt{\beta}$

We conclude that $\boxed{\lambda = \dfrac{X}{\sqrt{\Delta_1}\sqrt{\Delta_2}}}$ $\quad \left(\sqrt{\dfrac{X}{\Delta_1}} \cdot \sqrt{\dfrac{X}{\Delta_2}}\right)$

as desired.

(without needing to use integral w/ $ds = \sqrt{dx^2 + dy^2}/y$ over )

(5) **Cor:** $-\lambda(E) = \dfrac{\lambda(A)\lambda(C)}{\lambda(D)\lambda(B)}$ for



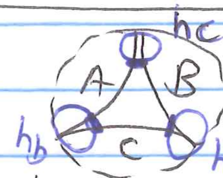
Note the sign

PF: $\lambda(A) = \dfrac{-x}{\sqrt{\Delta_d \Delta_a}}$, $\lambda(B) = \dfrac{1}{\sqrt{\Delta_a \Delta_b}}$

$\lambda(C) = \sqrt{\dfrac{y_c}{\Delta_b}}$, $\lambda(D) = \sqrt{\dfrac{y_c}{\Delta_d}}$.

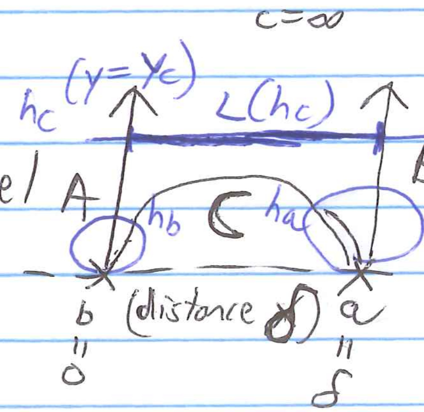
h_a, h_b, h_d have diameters $\Delta_a, \Delta_b, \Delta_d$

Cor: For a decorated ideal triangle let $L(h_c) :=$ hyperbolic length of arc segment along h_c between A & B



then $L(h_c) = \dfrac{\lambda(C)}{\lambda(A)\lambda(B)}$.

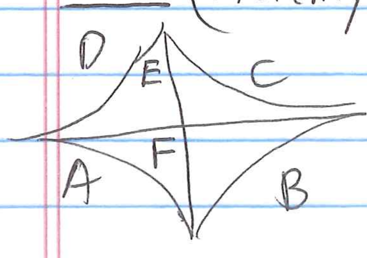
PF: Consider Upper-Half-Plane Model



$$L(h_c) = \int_{x=0}^{\delta} \frac{dx}{y_c} = \frac{x}{y_c}\Big|_0^{\delta} = \boxed{\frac{\delta}{y_c}}$$

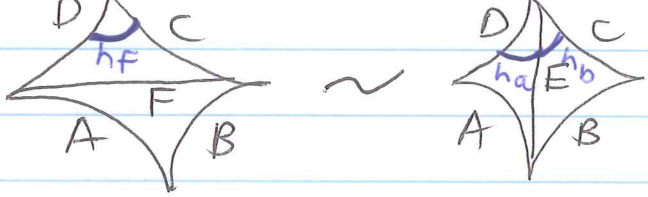
and from above $\dfrac{\lambda(C)}{\lambda(A)\lambda(B)} = \dfrac{\delta}{\sqrt{\Delta_b \Delta_a}} \Big/ \left(\sqrt{\dfrac{y_c}{\Delta_b}} \cdot \sqrt{\dfrac{y_c}{\Delta_a}}\right) = \dfrac{\delta}{y_c}$ ▨

Cor: (Ptolemy Lemma for Hyperbolic Quadrilateral)



$$\lambda(E) \cdot \lambda(F) = \lambda(A)\lambda(C) + \lambda(B)\lambda(D).$$

(6) Pf:

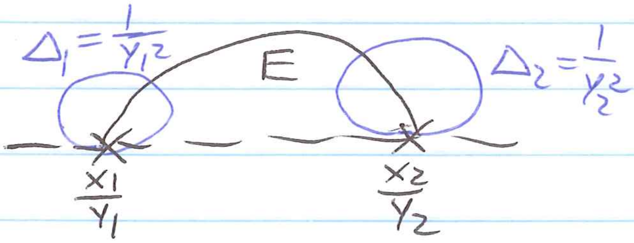


$$L(h_F) = L(h_a) + L(h_b)$$

$$\frac{\lambda(F)}{\lambda(D)\lambda(C)} = \frac{\lambda(A)}{\lambda(D)\lambda(E)} + \frac{\lambda(B)}{\lambda(E)\lambda(C)} \quad \blacksquare$$

Can also encode choice of ideal pt in $\mathbb{RP}^1$ plus a horocycle as $(x,y) \in \mathbb{R}^2 \longmapsto x/y$, $\Delta = 1/y^2$.

Claim: Under this correspondence

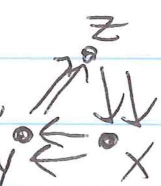


$$\boxed{\lambda_E = \left|\det\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}\right|}.$$

Pf: $$\lambda_E = \frac{\left(\frac{x_2}{y_2} - \frac{x_1}{y_1}\right)}{\sqrt{\frac{1}{y_1^2}} \cdot \sqrt{\frac{1}{y_2^2}}}$$ $\blacksquare$

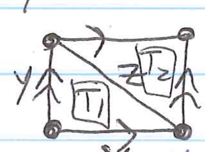
Yields: Direct translation between Ptolemy Relation and Plücker Relation for $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$.

Cor: For Markoff Cl. Alg.,

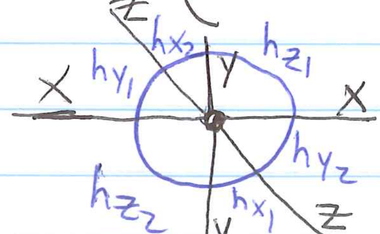


$$MI = \frac{x^2+y^2+z^2}{xyz}$$ invariant in all clusters (as from HW 1).

Pf:



$\rightsquigarrow$ around the unique puncture



h_p = full horocycle

$\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} + \cdots$ $\blacksquare$

$$L(h_p) = L(h_{x_1}) + L(h_{y_2}) + L(h_{z_1}) + L(h_{x_2}) + L(h_{y_1}) + L(z_2) = 2 \cdot MI.$$

identifying λ-length w/ associated edge.