10/17/18

**Thm (William Thurston)** For a fixed triangulation $T$ without self-folded triangles, the map

$$L \longrightarrow \{ b_L(E_i ; T) \}_{E_i \in T} \quad \text{is a } \underline{\text{bijection}}$$

$$\text{integral unbounded measured laminations} \longleftrightarrow \mathbb{Z}^n$$

In fact, $b_L(E_i ; T)$'s transform like a row of exchange matrix under mutation.

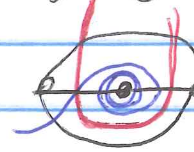
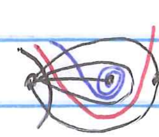
$\Rightarrow$ **Thm (W. Thurston, Fock-Goncharov)** If $T$ & $T'$ are triangulations of $(S,M)$ w/o self-folded triangles related to one another by flipping $E_K$.

Then $\tilde{B}(T, \{L_j\}_{j=1}^{m})$ & $\tilde{B}(T', \{L_j\}_{j=1}^{m})$ related by $\mu_K$

where $\tilde{B}(T, L) := \left[\begin{array}{c} B(T) \\ \hline B(L_j)_{j=1}^{m} \end{array}\right] = \left[\begin{array}{c} B(T) \\ \hline B(L_1) \\ B(L_2) \\ \vdots \\ B(L_m) \end{array}\right]$

for a choice of $m$ different laminations

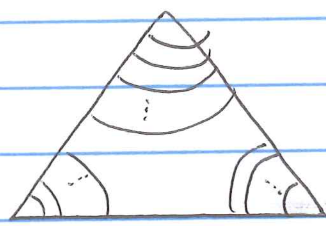
Each $L_j \longmapsto$ row $[ b_{L_j}(E_1 ; T), \dots, b_{L_j}(E_n ; T) ]$

Technical note: Can also extend def'n to  or  but we omit the proof

# Proof of Thurston's Thm

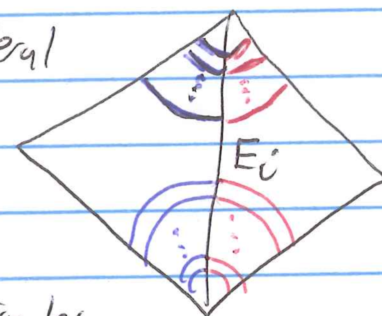
Given vector $\vec{b} = (b_{E_1}, b_{E_2}, \dots, b_{E_n}) \in \mathbb{Z}^n$
for triangulation $T = \{E_1, E_2, \dots, E_n\}$
with no self-folded triangles

Draw



in each triangle of $T$
where we have infinitely many arc segments between pairs of adjacent sides.

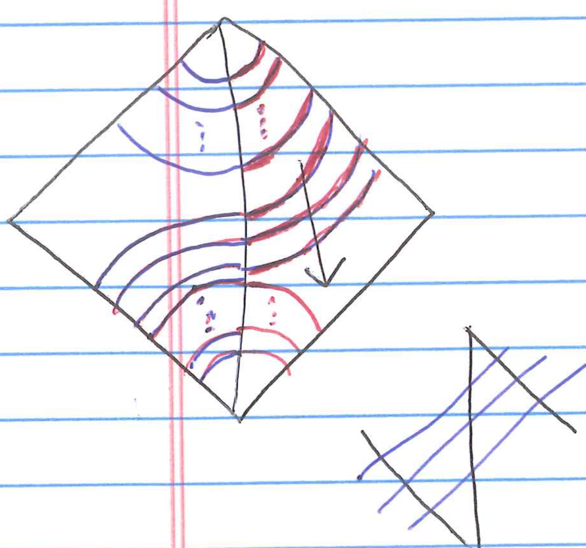
Given $E_i$ inscribed in quadrilateral



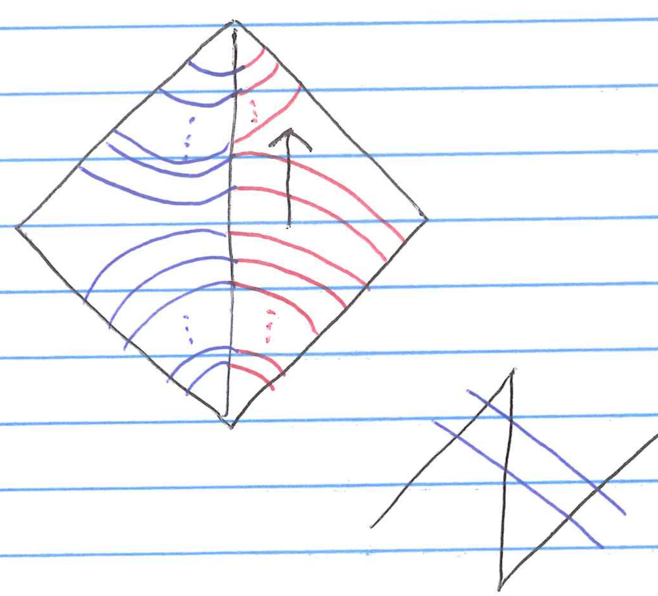

Case of $b_i = 0$

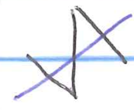
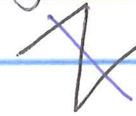
let $b_i \in \mathbb{Z}$ denote the shear/shift when connecting together the configurations on adjacent triangles

e.g. $b_i = +3$



$b_i = -2$



③ After shifting/shearing along each internal arc have local  or  configurations as indicated by $\vec{b} \in \mathbb{Z}^n$

<u>plus</u> <u>infinitely</u> many non-contributing curves traversing adjacent sides.

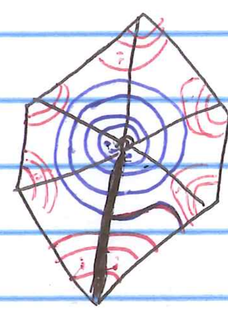
Individual curves (after triangles of $T$ all glued together) will either

- <u>travel from boundary to boundary</u>
- <u>connect up as a closed curve</u>

or

- <u>spiral into a puncture at one or both ends</u>

In latter case, we see e.g.




if $\vec{b} = [0, 0, \ldots, \overset{E_i}{-1}, 0, 0, \ldots, 0]$

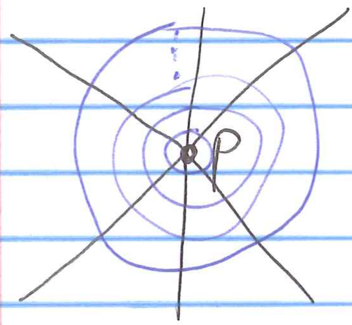
where shifting connections along $E_i$ one unit down while keeping all other connections as concentric circles,

leads to configuration <u>spiralling inward clockwise</u> to puncture

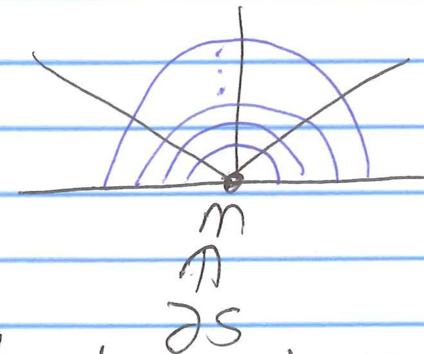
$\vec{b} = [0, \ldots, 0, +1, 0, \ldots, 0]$ leads analogously to <u>spiralling counter-clockwise into puncture.</u>

<u>Note:</u> To get bijection (i.e. unique $L$) we remove infinitely many copies of disallowed non-contributing elem. laminations

④ Since $b_i$'s are finite, while there are infinitely many curves around a single marked point in $(S, M)$, e.g.

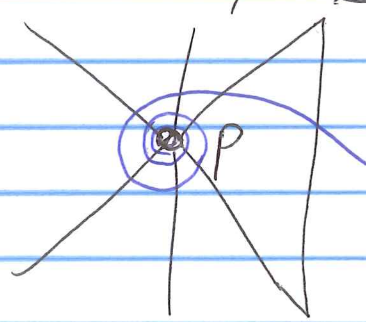


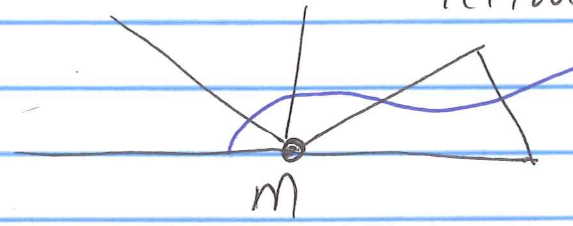

or




all of these are <u>disallowed</u> and can be removed from the lamination.

Leaves only a <u>finite number of allowed</u> elem. laminations leftover.

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<u>Note:</u> Also a contractible curve <u>never</u> arises from this construction.


or


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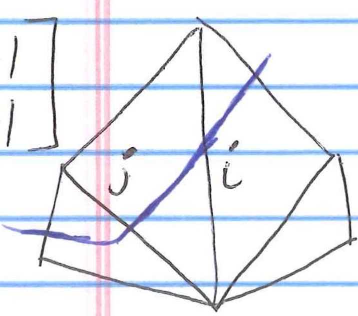
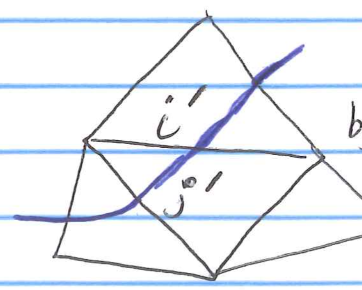
If we flip/mutate a triangulation

$$\begin{array}{c} b_i \\ b_j \end{array} \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$




$$\xrightarrow{\mu_i} \quad \boxed{b_{ij} = +1}$$




$$\begin{array}{c} b_i' \\ b_j' \end{array} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$0 = -1 + [+1]_+$

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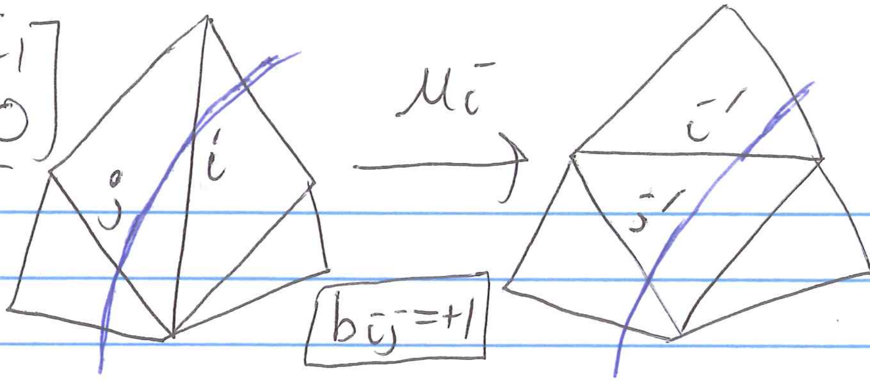
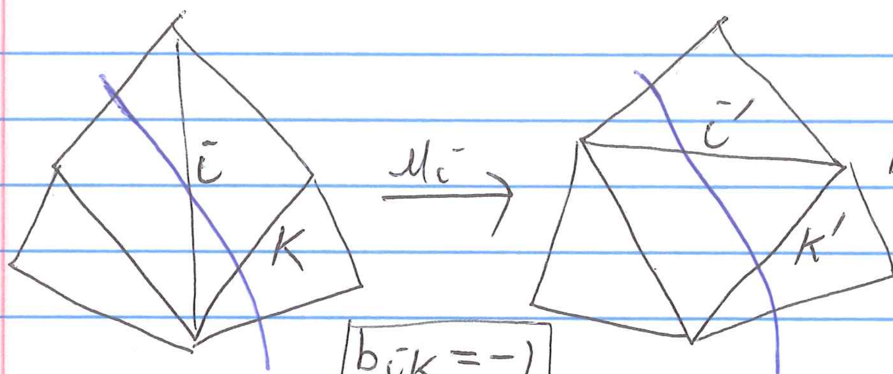
$$\begin{array}{c} b_i \\ b_j \end{array}\begin{bmatrix} +1 \\ 0 \end{bmatrix}$$




$\boxed{b_{ij} = +1}$

$$\begin{array}{c} b_i' \\ b_j' \end{array}\begin{bmatrix} -1 \\ +1 \end{bmatrix}$$
$$\| $$
$$0 + [+1]_+$$

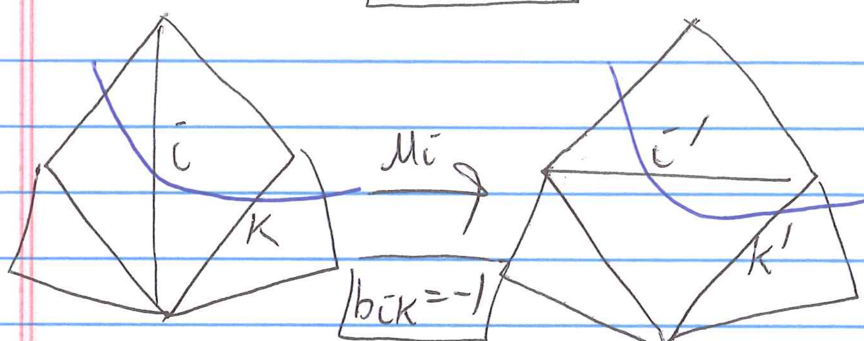
$$\begin{array}{c} b_i \\ b_k \end{array}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$




$\boxed{b_{ik} = -1}$

$$\begin{array}{c} b_i' \\ b_k' \end{array}\begin{bmatrix} +1 \\ -1 \end{bmatrix}$$
$$\|$$
$$0 - [-(-1)]_+$$

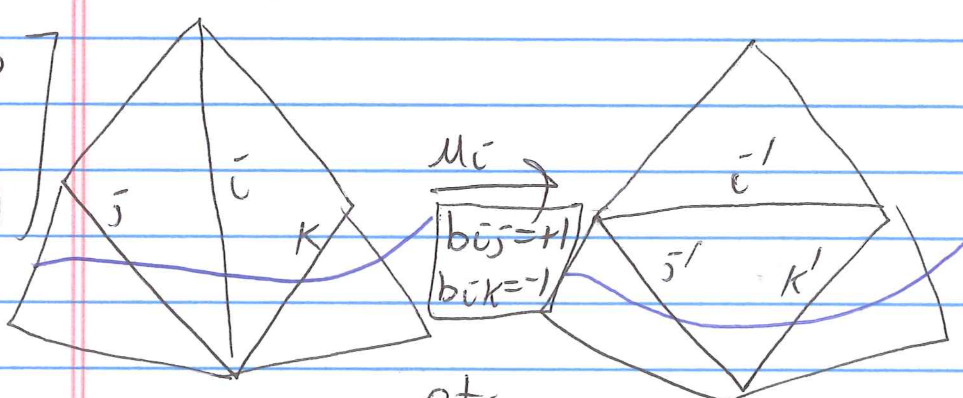
$$\begin{array}{c} b_i \\ b_k \end{array}\begin{bmatrix} -1 \\ +1 \end{bmatrix}$$




$\boxed{b_{ik} = -1}$

$$\begin{array}{c} b_i' \\ b_k' \end{array}\begin{bmatrix} +1 \\ 0 \end{bmatrix}$$
$$\|$$
$$1 - [-(-1)]_+$$

$$\begin{array}{c} b_i \\ b_j \\ b_k \end{array}\begin{bmatrix} 0 \\ -1 \\ +1 \end{bmatrix}$$




$\boxed{\begin{array}{l} b_{ij} = +1 \\ b_{ik} = -1 \end{array}}$

$$\begin{array}{c} b_i' \\ b_j' \\ b_k' \end{array}\begin{bmatrix} 0 \\ -1 = -1 + [0]_+ \\ +1 = +1 - [-0]_+ \end{bmatrix}$$

etc.

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$b_i' = -b_i$ & for $j \neq i$)

$\left( b_{ij} = \text{usual exchange matrix entry} \right.$

$\left. \max(\alpha, 0) = [\alpha]_+ \right)$

$b_j' = b_j + \max(b_{ij}, 0)$ if $b_{ij} > 0$

$b_j' = b_j - \max(-b_{ij}, 0)$ if $b_{ij} < 0$

$b_j' = b_j$ if $b_{ij} = 0$

⑥ **Thm** : Given extended cluster $\{x_1, x_2, \dots, x_{n+c}\}$
converted to $\tau$-coordinates $\{\tau_1, \tau_2, \dots, \tau_{n+c}\}$

Mutating $x_i \longrightarrow x_i'$ changes $\tau$-coords by

$$\tau_i' = 1/\tau_i$$

exchange matrix entry

$$\tau_j' = \begin{cases} \tau_j (1+\tau_i)^{b_{ij}} & \text{if } b_{ij} > 0 \\ \tau_j \left(1+\frac{1}{\tau_i}\right)^{b_{ij}} & \text{if } b_{ij} < 0 \\ \tau_j & \text{if } b_{ij} = 0 \end{cases}$$

Cross ratio coords $x_j$'s transform in the same exact way

and $b_L(E_j; T)$'s transform by tropical version

$$b_L(E_i'; T') = -b_L(E_i; T)$$

$$b_L(E_j'; T') = \begin{cases} b_L(E_j; T) + b_{ij}\left[b_L(E_i; T)\right]_+ & \text{if } b_{ij} > 0 \\ b_L(E_j; T) + b_{ij}\left[-b_L(E_i; T)\right]_+ & \text{if } b_{ij} < 0 \\ b_L(E_j; T) & \text{if } b_{ij} = 0 \end{cases}$$

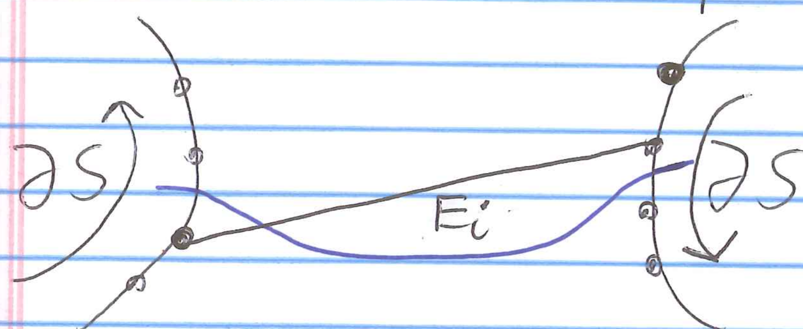
(where $[\alpha]_+ = \max(\alpha, 0)$
$= (1 \oplus \alpha)$

$[-\alpha]_+ = (1 \oplus -\alpha)$)

⑦ <u>Principal coefficients</u> For any cluster alg. from a surface, can build <u>$2n$-by-$n$ extended exch. matrix</u>

by $$\tilde{B} = \begin{array}{c} \begin{bmatrix} B \\ \hline I \end{bmatrix} \end{array} \begin{array}{l} E_1 \\ \vdots \\ E_n \\ L_1 \\ \vdots \\ L_n \end{array}$$ where $L_i$ defined by arc $E_i$ as follows

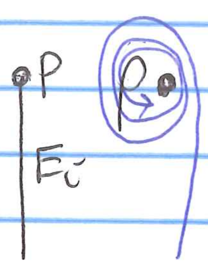
If $E_i$ goes from marked points on boundary to boundary



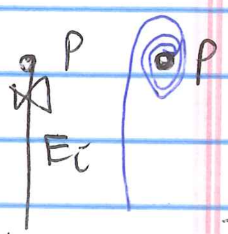

we define <u>$L_i$ as elem. lamination</u> by shifting endpoints of $E_i$ by $\varepsilon$ on both sides in the "positive" direction.

(counter-clockwise if $\partial S$ component a bounded region)

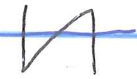
If $E_i$ connects to a puncture on <u>one or both ends</u> <u>spiral counter-clockwise</u> into it instead




If $E_i$ " " but is notched/tagged <u>spiral clockwise</u> into it instead




<u>Claim</u>: Such choices of $L_1, \ldots, L_n$ yields above $\tilde{B}$.

<u>PF</u>: We see local  configs as desired.