11/28/18 Let Δ be a lattice polygon in $\mathbb{R}^2$, all of whose vertices lie in $\mathbb{Z}^2$.

There is an action of $SL_2\mathbb{Z} \ltimes \mathbb{Z}^2$ on the set of such lattice polygons: $SL_2\mathbb{Z}$ acts by shearing, i.e. a basis change with change-of-basis-matrix of determinant $+1$, and $\mathbb{Z}^2$ acts by affine translation.

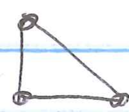
S

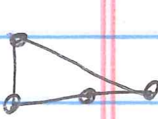
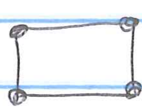
Such an action preserves the area of Δ, the number of interior points, the length of its sides, and the number of sides of Δ.

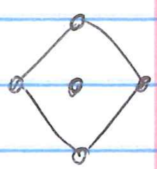
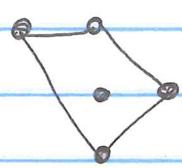
I

Let $B = \#$ exterior, i.e. boundary pts of Δ.

By Pick's Theorem $S = I + \frac{B}{2} - 1$.

E.g.'s  has $S=\frac{1}{2}$, $I=0$, $B=3$

 or  has $S=1$, $I=0$, $B=4$

 or  has $S=2$, $I=1$, $B=4$

Claim: By a construction of Goncharov-Kenyon, from Δ, we can build a $\mathcal{X}$-cluster Poisson variety $\mathcal{X}$ whose dimension is $\dim \mathcal{X} = 2S$. Uses a quiver Q_Δ with $2S$ vertices.

Note: since $\text{Area}(\Delta) = 1/2$ $S \in \mathbb{Z}/2$, $2S$ odd or even integer.

11/28/18 ② Q_Δ will have ε_{ij} arrows from i to j where

$\mathcal{X}$ is equipped with Poisson bracket $\{X_i, X_j\} = \varepsilon_{ij} X_i X_j$.

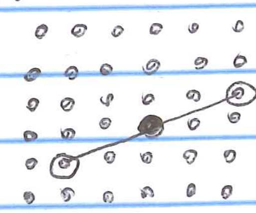
As before, $\prod_{i=1}^{25} X_i = 1$.

<u>Goncharov-Kenyon's construction</u>: Orient $\partial\Delta$, e.g. counterclockwise. Let $e_1, e_2, \dots, e_B \in \mathbb{Z}^2$ denote the <u>primitive vectors</u> associated to each side of Δ.

Here <u>primitive</u> means if $e_i = \begin{bmatrix} a \\ b \end{bmatrix}$ then $\gcd(a,b) = 1$.
Otherwise if we have a side w/ corners (c,d) & (f,g) s.t. $(f,g) - (c,d) = \begin{bmatrix} Da \\ Db \end{bmatrix}$ with $D = \gcd(f-c, g-d) > 1$

then this side would have length D and contain $D+1$ collinear points

E.g. $(5,3) - (1,1) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$



In the collinear case, we repeat e_i multiple times in a row in our collection of B primitive vectors.

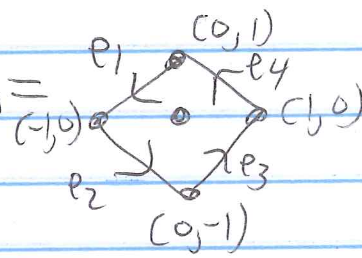
We will mostly focus on the case w/ no multiplicities.

By construction, $\sum e_i = \vec{0}$.

A torus T can be represented as $\mathbb{R}^2/\mathbb{Z}^2$ so that each vector e_i determines a homology class $[e_i] \in H_1(T, \mathbb{Z})$.
In fact $\exists!$ geodesic (line in $\mathbb{R}^2/\mathbb{Z}^2$) of direction e_i.

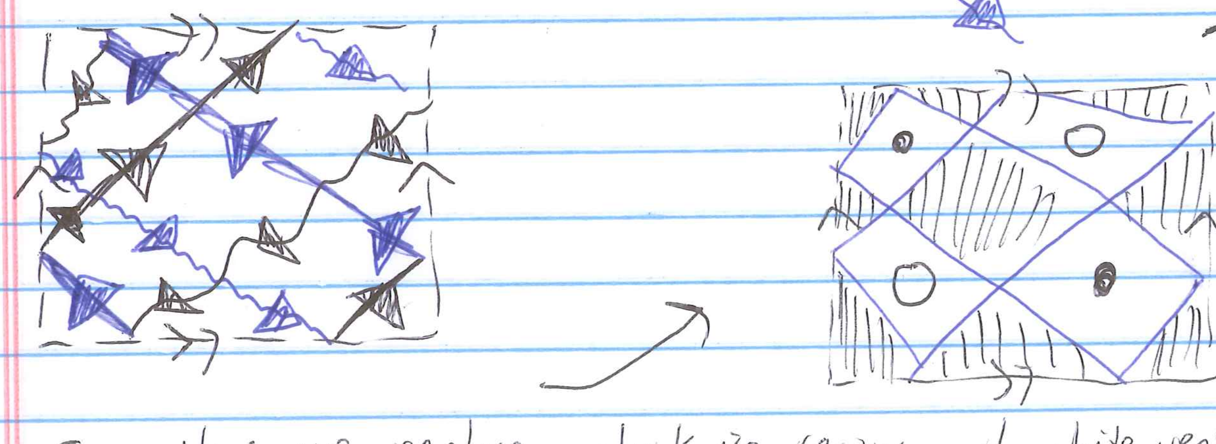
(3) Up to translation, place loops $\gamma_1, \gamma_2, \ldots, \gamma_B$ on T (lines in $\mathbb{R}^2/\mathbb{Z}^2$) oriented in the <u>direction of e_i</u> for <u>γ_i</u>

s.t.
- No triple intersections
- Total Number of intersection points is minimal
- <u>Alternating Strand Condition</u>: Following loop γ_j, we encounter loops $\gamma_{j_1}, \gamma_{j_2}, \ldots, \gamma_{j_d}$ so they alternate crossing γ_i from left-to-right to right-to left.

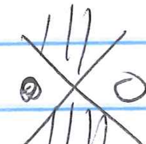
<u>Example</u>: $\Delta =$ 

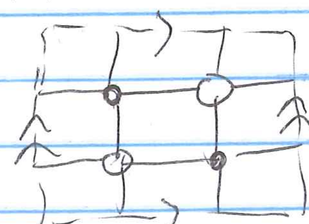
$e_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $e_4 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$

$e_2 = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$, $e_3 = \begin{bmatrix} +1 \\ +1 \end{bmatrix}$



<u>Secondly</u>: we replace clockwise regions w/ white vertices
" " counter-clockwise " " black " "
and contract alternating regions (shade first)

<u>Thirdly</u>: Replace  with 

Yields our familiar  in this case.

Call this bipartite graph Γ_Δ on torus.

(4) Dualizing (i.e. faces $\longleftrightarrow$ quiver vertices, edges $\longleftrightarrow$ arrows) yields Q_Δ.

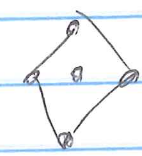
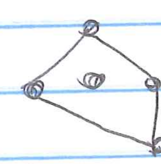
We get an X_i for each vertex of Q_Δ,
i.e for every face of Γ_Δ
which is counted by the normalized area $\underline{2S}$ of Δ.

E.g. $2S = 4$ & $Q = $
$$\begin{tikzcd} \circ \arrow[r, shift left] \arrow[r, shift right] & \circ \arrow[d, shift left] \arrow[d, shift right] \\ \circ \arrow[u, shift left] \arrow[u, shift right] & \circ \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$
in the running example.

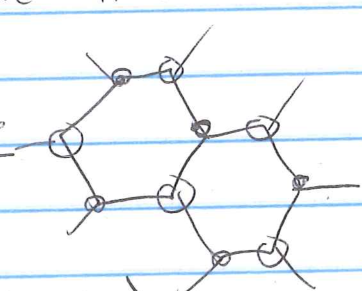
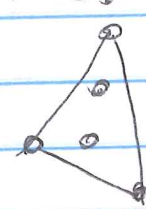
We build the conjugate surface by switching cyclic orientation around white vertices,

i.e. zig-zags on $\Gamma \longrightarrow$ faces on $\hat{S}$

Claim: Genus of conjugate surface $\hat{S}$ is I, the number of interior lattice points of Δ.

In two of our running examples, e.g.  & 

$I = 1$ and we indeed saw $\hat{S}$ was again a torus.

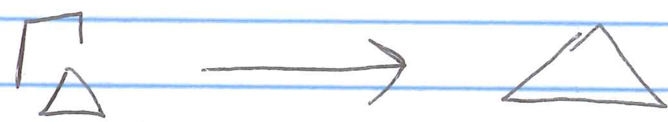
Kenyon Slide e.g.  yields $\Delta = $ 

with $I = 2$
and indeed genus$(\hat{S}) = 2$
in such a case.

(see Nov. 21 notes)

(5) We now can be more precise than on Monday:

Claim: If we build Kasteleyn matrix ~~K~~ from Γ let $P_\Gamma(z_1, z_2) = \det K_\Gamma$ with all edge weights $= 1$, then the Newton polygon of $P_\Gamma(z_1, z_2)$ sends



bipartite graph on a torus

up to the action of $SL_2\mathbb{Z} \ltimes \mathbb{Z}^2$

[recover the lattice polygon used in the construction] except possibly for multiplicities on internal vertices

Leads to a natural question we discuss on Friday:

Ques: If two bipartite graphs Γ_1 and Γ_2 yield the same Newton Polygon as above, how are Γ_1 and Γ_2 related?

But first: Let us prove for Planar bipartite graphs that a Kastelyn weighting leads to det K with all terms of the same sign. Then Proof can be adapted to the higher genus case.

Assume Γ is a bipartite planar graph such that the number of black vertices equals the number of white vertices. Let n = common number. Kastelyn Matrix K is $n \times n$.

(6) Let $\tilde{\Gamma}$ be the complete bipartite graph $K_{n,n}$ with weights on edges agreeing w/ weights of Γ except $w(b_i, w_j) = 0 \iff \Gamma$ has no edge $\underset{b_i}{\bullet}$ $\underset{w_j}{\circ}$.

Clear that partition function $Z_\Gamma = \sum_{\substack{M \text{ perfect} \\ \text{matching of } \Gamma}} \prod_{e \in M} w(e)$

satisfies $Z_\Gamma = Z_{\tilde{\Gamma}}$.

By construction, $\det K_{\tilde{\Gamma}} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) K(b_1, w_{\sigma(1)}) \cdots K(b_n, w_{\sigma(n)})$

where $K(b_i, w_j) = \pm w(b_i, w_j)$ w/ sign given by Kastelyn weighting.

Since σ is a permutation, i.e. a bijection, and $w(b_i, w_j) \neq 0 \iff \Gamma$ has edge $\underset{b_i}{\bullet}\!\!-\!\!\!-\!\!\underset{w_j}{\circ}$, it follows that each term of $\det K_\Gamma$ is indeed a perfect matching of Γ,

To show $Z_\Gamma = \underbrace{\pm}_{\text{global sign}} (\det K_\Gamma)$, it thus suffices to show all non-zero terms have the same sign.

Rem: Above arguments true for any bipartite Γ. We now use planarity.

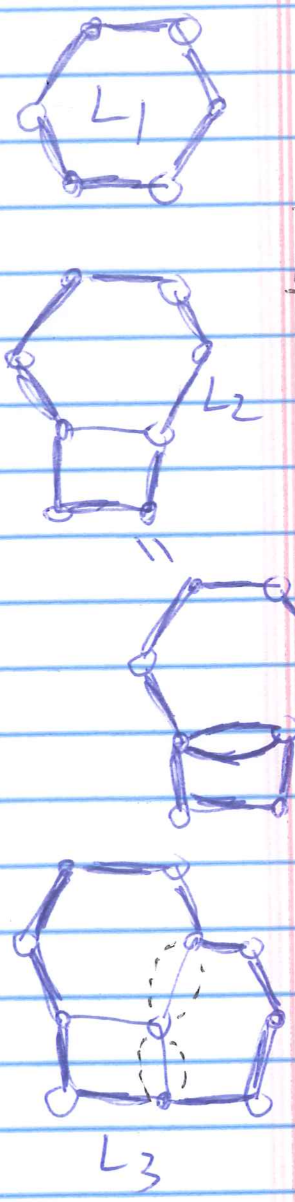
Superimpose matchings M_1 and M_2 on Γ.

$M_1 \cup M_2$ can be decomposed into
- connected components of loops of size ≥ 4

OR
- doubled edges

Lemma: For a planar graph, given a Kastelyn weighting (even # -1's on $(4k+2)$-gon, odd # -1's on $4k$-gon), and a $2m$-cycle L $(m \geq 2)$ enclosing l vertices

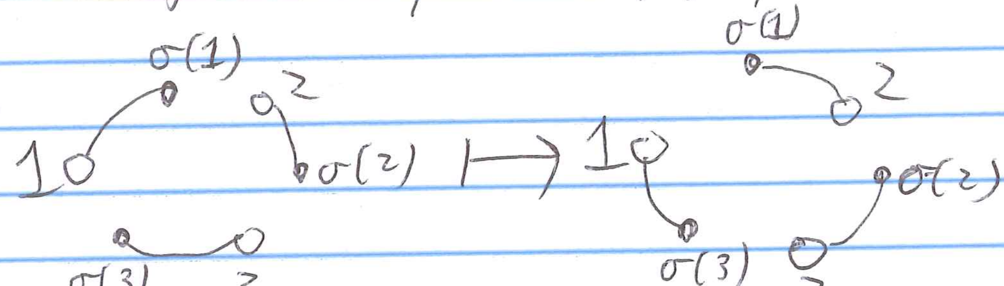
then $\boxed{\text{the alternating product of Kastelyn weights is } (-1)^{m+l+1}}$.



Pf of Lemma by Induction on # faces of Γ enclosed by L.

Each local change on a loop L of size $2m$ also changes σ by multiplying by a m-cycle

e.g.



has $\sigma'(1) = \sigma(3)$, $\sigma'(2) = \sigma(1)$, $\sigma'(3) = \sigma(2)$.

The # interior vertices enclosed by L, i.e. l, must be even since they would be grouped together into even size loops or doubled edges as well.

$\Rightarrow$ each local change by L alters alternating product of Kastelyn weights by $(-1)^{m+1}$ = sgn of an m-cycle.

$\Rightarrow$ all perfect matchings of a planar graph (all equiv. by local moves) yield the same sign for its product of Kastelyn weights.