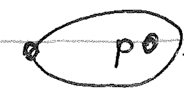
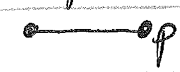
Cluster algebras from Surfaces Positivity

Thm [MSW] The Laurent expansion of cluster variable x_γ, corresponding to an ordinary arc γ is given by

if with principal coeffs

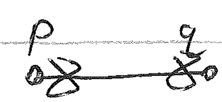
$$x_\gamma = \sum_{\substack{\text{perfect matchings}\\ M \text{ of snake graph } G_\gamma}} x(M)\, y(M) \Big/ x_{i_1} x_{i_2} \cdots x_{i_k}$$

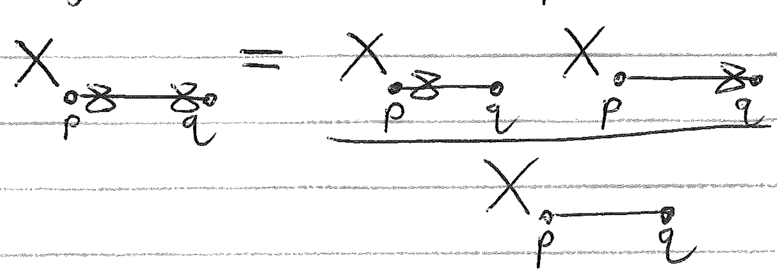
if γ crosses initial triangulation in arcs $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_k}$ (possibly w/ multiplicities) in order.

Thm: For a tagged arc γ_p, singly-notched at puncture p, i.e. $\gamma_p =$ , then let $\ell_p =$ once punctured monogon, treated as an ordinary arc.  ℓ_p, $r_p =$ untagged γ_p 

$$\boxed{x_{\gamma_p} := x_{\ell_p} / x_{r_p}}$$

There is also a positive gen function without division with "symmetric" perfect matchings (i.e. equiv. classes) as summands.

Thm: For doubly tagged arcs γ_{pq} , analogous but more complicated method using identity



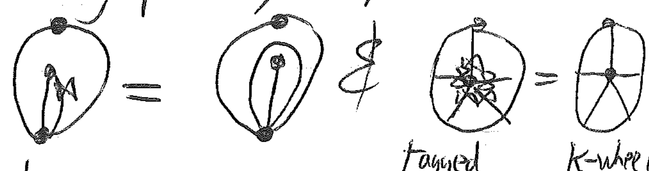
if y_i's $= 1$.

See [MSW, Thm 12.9] with principal coeffs.

Sketch of Proof (for ordinary arcs) see Section 6 of [MSW11]

- Proof for Type A_n (Polygon) case

- Any tagged triangulation can be replaced by an ideal triangulation, possibly w/ self-folded triangles, by



under the tagged K-wheel → K-wheel,
when computing X_γ, also reverse tagging at puncture for γ also.

- Essentially lift X_γ crossing (part of) an ideal triangulation T to a cover, e.g. universal cover in unpunctured case, and then can treat lifts $\tilde{\gamma}$ crossing a polygon in lifted $\tilde{T}$.

- Apply an algebraic specialization, i.e. homomorphism of cluster algebras, to get desired result downstairs.

- Show that specialization compatible w/ the desired relations.

- Since wanted proof in case w/ principal coeffs, a lot of the work involved "laminations" and the y_i's.

- "Quadrilateral Lemma" showing that induction order still makes sense even w/ arcs in more complicated surfaces where arcs cross arcs of triangulation multiple times. (i.e. can find simply-connected quadrilateral inscribing arc where crossing #'s monotonicly less on boundary)

(3)

More details for Type A_n case

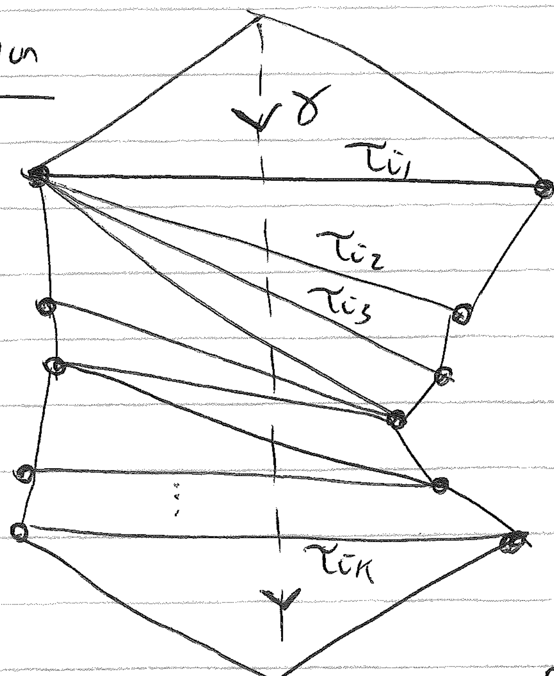
[Adaption of [Schiffler '08] arXiv:0611956]

Consider a triangulation T of an $(n+3)$-gon. Let γ be an arc not in T.

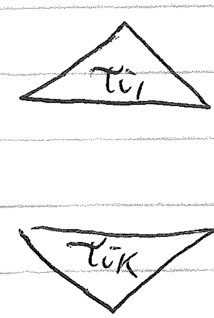
<u>Claim:</u> $$X_\gamma = \sum_{\substack{M \text{ perf. matching} \\ \text{of snake graph } G_\gamma}} x(M) \Big/ x_{i_1} x_{i_2} \cdots x_{i_k}$$

where γ crosses $\tau_{i_1}, \tau_{i_2}, \tau_{i_3}, \ldots, \tau_{i_k}$ in order (all distinct in this case)

<u>Illustration</u>

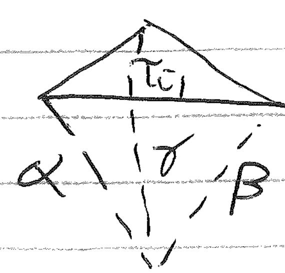


T always contains



PF: By induction on K. Build quadrilateral $Q_{\gamma,\tau_{i_1}}$ whose two diagonals are γ & τ_{i_1}.
(Generically, $Q_{\gamma,\tau_{i_1}}$ will not be part of triangulation T, but that's o.k.)

e.g. $Q_{\gamma,\tau_{i_1}} =$



(4) Arcs α & β, the other two boundaries of $Q_{\gamma,\tau_{i_1}}$ may or may not be part of T.

But α & β both cross T fewer times than γ did since neither cross τ_{i_1} and at worst cross $\tau_{i_2}, \tau_{i_3}, \ldots, \tau_{i_k}$.

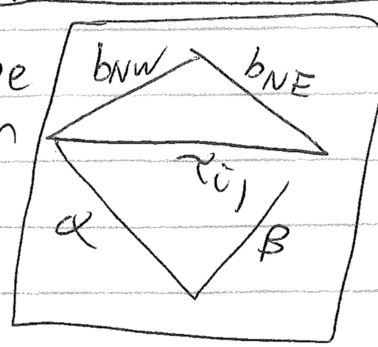
So by induction, $$x_\alpha = \sum_{\substack{M_\alpha \text{ P.M.} \\ \text{of } G_\alpha}} x(M_\alpha)/* \;\&$$

$$x_\beta = \sum_{\substack{M_\beta \text{ P.M.} \\ \text{of } G_\beta}} x(M_\beta)/*$$

Since there is a reachable cluster corresponding to a triangulation T' with $Q_{\gamma,\tau_{i_1}}$ inside of it,

we get $$\boxed{x_{\tau_{i_1}} x_\gamma = x_{b_{NW}} x_\beta + x_{b_{NE}} x_\alpha}$$

where we let b_{NW} & b_{NE} be as in



Based on the construction of snake graphs

G_β (W/o loss of generality assume β crosses $\tau_{i_2}, \ldots, \tau_{i_k}$) looks identical to G_γ except w/o tile $\boxed{\tau_{i_1}}$.

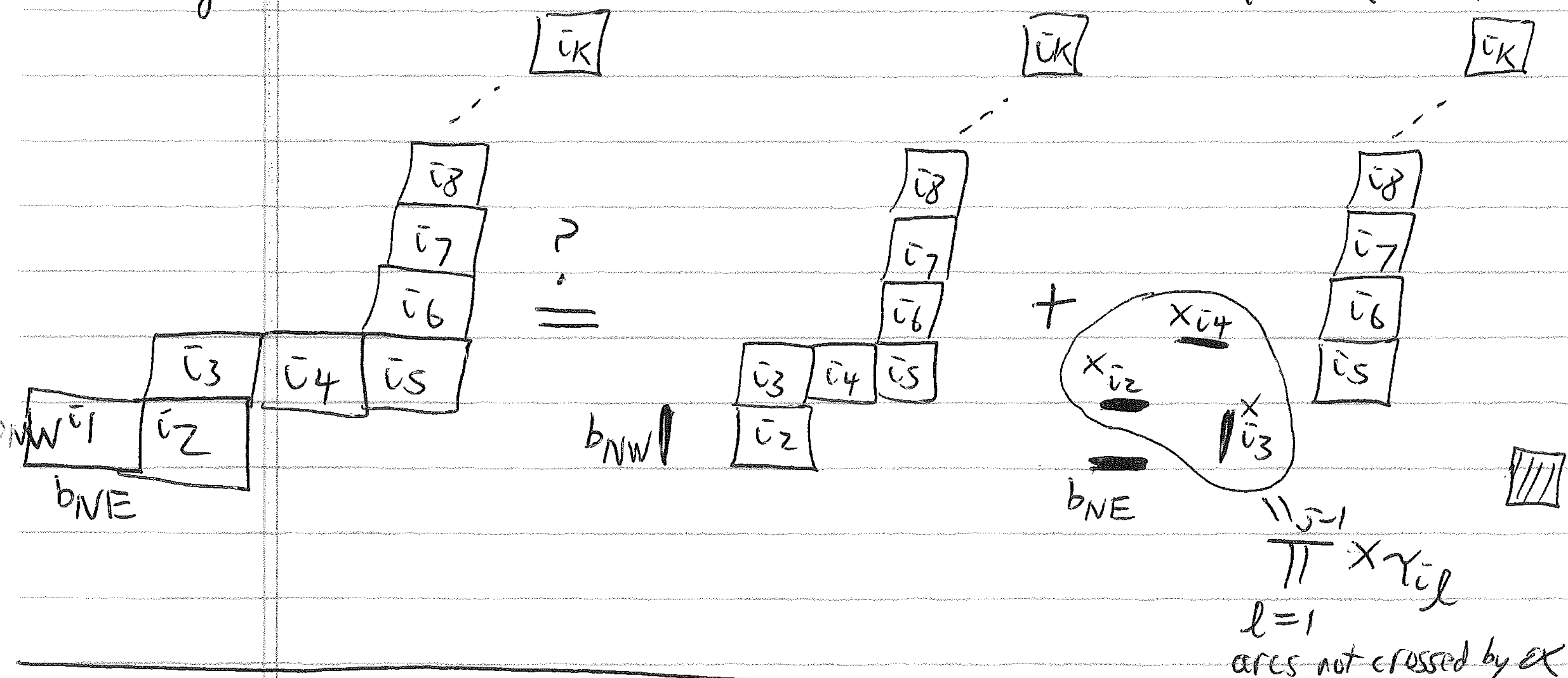
and there exists $j \geq 3$ so α crosses $\tau_{i_j}, \tau_{i_{j+1}}, \tau_{i_{j+2}}, \ldots, \tau_{i_k}$ & G_α is also a subgraph of G_γ.

(5) To verify the inductive step, need to check

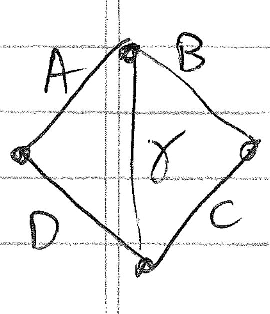
$$x_{\tau_{i_1}}\left(\frac{\sum_{M \text{ in } G_\gamma} x(M)}{x_{\tau_{i_1}} x_{\tau_{i_2}} \cdots x_{\tau_{i_k}}}\right) \overset{?}{=} x_{b_{NW}}\left(\frac{\sum_{M_\beta \text{ in } G_\beta} x(M_\beta)}{x_{\tau_{i_2}} x_{\tau_{i_3}} \cdots x_{\tau_{i_k}}}\right) + x_{b_{NE}}\left(\frac{\sum_{M_\alpha \text{ in } G_\alpha} x(M_\alpha)}{x_{\tau_{i_j}} x_{\tau_{i_{j+1}}} \cdots x_{\tau_{i_k}}}\right)$$

Multiplying through by denominators,

$$\sum_{M \text{ in } G_\gamma} x(M) \overset{?}{=} x_{b_{NW}}\left(\sum_{M_\beta \text{ in } G_\beta} x(M_\beta)\right) + x_{b_{NE}}\left(\sum_{M_\alpha \text{ in } G_\alpha} x(M_\alpha)\right)\prod_{l=1}^{j-1} x_{\tau_{i_l}}$$



Quadrilateral Lemma: Case-by-case analysis shows that even for more complicated surfaces and ideal triangulations, one can find an inscribed quadrilateral in (S, M).

$Q_\gamma =$ 

w/ A, B, C, D may or may not be in T such Q_γ simply-connected & A, B, C, D each cross T less times than γ. See [MSW11, Sec. 9].