Cluster algebras from Surfaces Positivity

Thm \([MSW]\) The Laurent expansion of cluster variable \(X_y\) corresponding to an ordinary arc \(y\) is given by

\[
X_y = \sum_{\text{perfect matchings } M \text{ of snake graph } G_y} \frac{X(M) y(M)}{X_{i_1} X_{i_2} \ldots X_{i_K}}
\]

if \(y\) crosses initial triangulation in arcs \(\tau_{i_1} \tau_{i_2} \ldots \tau_{i_K}\) (possibly w/ multiplicities) in order.

Thm: For a tagged arc \(\gamma_p\) singly-notched at puncture \(p\), i.e. \(\gamma_p = \circ \frac{p}{p} \), then let \(l_p = \) once punctured monogon treated as an ordinary arc.

\[
X_{\gamma_p} = X_{l_p} / X_{\gamma_p}
\]

There is also a positive gen function without division with "symmetric" perfect matchings (i.e. equiv. classes) as syzygy.

Thm: For doubly tagged arcs \(\gamma_{pq}\) analogous but more complicated method using identity

\[
X_{\gamma_{pq}} = X_{\gamma_{rp}} X_{\gamma_{sq}} \quad \text{if } \gamma_{rs}'s = 1
\]

See \([MSW\text{ Thm 12.9]}\) with principal coeffs.
Sketch of Proof (for ordinary arcs) of [Msw1]

- Proof for Type A_n (Polygon) case

- Any tagged triangulation can be replaced by an ideal triangulation possibly w/ self-folded triangles, by

\[ \begin{align*}
\text{tagged} & \quad \Rightarrow \\
\text{under the tagged } k\text{-wheel } & \Rightarrow k\text{-wheel}
\end{align*} \]

when computing \( X \), also reverse tagging at puncture for \( X \) as well.

- Essentially lift \( X \) crossing (part of) an ideal triangulation \( T \) to a cover, e.g. universal cover in unpunctured case, and then can treat lifts \( \tilde{X} \) crossing a polygon in lifted \( \tilde{T} \).

- Apply an algebraic specialization, i.e., homomorphism, of cluster algebras, to get desired result downstream.

- Show that specialization compatible w/ the desired relations.

- Since wanted proof in case w/ principal coeffs, a lot of the work involved laminations and the \( \gamma_i \)'s.

- "Quadrilateral Lemma" showing that induction order still makes sense even w/ arcs in more complicated surfaces where arcs cross triangulation multiple times, i.e. can find quadrilateral inscribing arc where crossing #’s monodromy less on boundary.
More details for Type An case [Adoption of Schiffer '08]

Consider a triangulation $T$ of an $(n+3)$-gon. Let $\gamma$ be an arc not in $T$.

Claim: $\chi_{\gamma} = \sum \frac{x(M)}{x_{ij} \cdots x_{ik}}$

$M$ perf. matching of snake graph $G_\gamma$

where $\gamma$ crosses $T_{ij}, T_{iz}, T_{izj}, \ldots, T_{ik}$ in order
(all distinct in this case)

Illustration

$T$ always contains $T_{ij}$

$T_{ik}$

PF: By induction on $K$. Build quadrilateral $Q_{ij}$

whose two diagonals are $\gamma \notin T_{ij}$

(Generally $Q_{ij}$ will not be part of triangulation $T$

but that's O.K.)

e.g. $Q_{ij} = \frac{A}{\gamma} B$
Arcs $\alpha$ & $\beta$, the other two boundaries of $Q_{\delta_j, \tau_j}$ may or may not be part of $T$.

But $\alpha$ & $\beta$ both cross $T$ fewer times than $\gamma$ did since neither cross $T_{C_j}$ and at worst cross $T_{C_2}, T_{C_3}, \ldots, T_{C_k}$.

So by induction $X\alpha = \sum_{M_\alpha \text{ P.M.}} X(M)/*$ of $G_\alpha$

$X\beta = \sum_{M_\beta \text{ P.M.}} X(M)/*$ of $G_\beta$

Since there is a reachable cluster corresponding to a triangulation $T$ with $Q_{\delta_j, \tau_j}$ inside of it we get

$X_{\tau_j} \cdot \gamma = X_{bNW} X_\beta + X_{bNE} X_\alpha$

when we let $bNW$ & $bNE$ be as in

Based on the construction of snake graphs

$G_\beta$ (W/o loss of generality assume $\beta$ crosses $T_{C_2}, \ldots, T_{C_k}$) looks identical to $G_\gamma$ except w/o the

and here exists $j$ so $\alpha$ crosses $T_{C_3}, T_{C_{j+1}}, T_{C_{j+2}}, \ldots, T_{C_k}$ & $G_\alpha$ is also a subgraph of $G_\gamma$. 

\[\square\]
To verify the inductive step, we need to check

\[
\sum_{\text{Min } G} X_{\tau_i} \left( \frac{\sum \chi(M)}{X_{\tau_i} X_{\tau_{i+1}} \ldots X_{\tau_K}} \right) = X_{b_{NW}} \left( \sum_{\text{Min } G}\chi(M) \right)
\]


\[
+ X_{b_{NW}} \left( \sum_{\text{Max } G} \chi(M) \right)
\]


\[
\sum_{\text{Min } G} \chi(M) = X_{b_{NW}} \left( \sum_{\text{Min } G}\chi(M) \right) + X_{b_{NW}} \left( \sum_{\text{Max } G} \chi(M) \right) \prod_{l=1}^{s-1} X_{\tau_{i+l}}
\]

**Quadrilateral Lemma:** Case-by-case analysis shows that even for more complicated surfaces and ideal triangulations, one can find an inscribed quadrilateral in \((S, M)\).

\[Q_x = A B C D\]

\(Q_x\) is simply-connected and \(A, B, C, D\) each cross \(T\) less times than \(D\). See \([M, S, W, 1, 1, \text{ Sec. } 7]\).