11/21/18 Last time we discussed Moduli space $\mathcal{L}_\Gamma$

$$\mathcal{L}_\Gamma = \left\{ \mathbb{C}^*\text{- edge weights} \right\} \Big/ \left\{ \begin{array}{l} \text{gauge transformations} \\ \text{multiplying by } \alpha_v \text{ for} \\ \text{all edges incident to vertex } v \end{array} \right\}$$

for a bipartite graph Γ on a torus.

Γ has 2-cell complex structure w/ V, E, F.

Claim: $\dim \mathcal{L}_\Gamma = |F(\Gamma)| + 1$.

Pf: Consider the $|V| \times |E|$ $(0,1)$-matrix M_Γ s.t. $(M_\Gamma)_{ve} = \begin{cases} 1 & \text{if edge } e \text{ incident to vertex } v \\ 0 & \text{o.w.} \end{cases}$

Since Γ is a connected graph, $\boxed{\text{rank } M_\Gamma = |V| - 1}$

To see this, note that since Γ is bipartite, the vector δ_Γ whose entries are $+1$ on black vertices, -1 on white vertices is in $\operatorname{Ker} M_\Gamma$. (as left multiplication)

Because Γ is connected, we claim that any other element of $\operatorname{Ker} M_\Gamma$ is a scalar multiple of δ_Γ.

Let $[z_1, \dots, z_{|V|}]$ be in (left) $\operatorname{Ker} M_\Gamma$. Columns of M_Γ have two 1's each so $\vec{z} M_\Gamma = [\underbrace{z_{b_1} + z_{w_1}, z_{b_2} + z_{w_2}, \dots}_{|E|}]$ where (b_i, w_i) are the black & white endpoints of edge e_i.

Hence $z_v = -z_w$ whenever they are endpoints of an edge, Γ connected and bipartite so $z_v = (-1)^l z_w$ if path of length l $v \rightsquigarrow w$

Since rank $M_\Gamma = |V|-1$, there are this many algebraically independent choices of $\{\alpha_v : v \in V\}$ when scaling edge weights.

Hence $\dim \mathcal{L}_\Gamma = |E| - (|V|-1) = |E| - |V| + 1$.

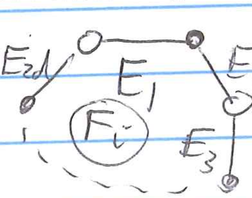
Finally, since Γ is on a torus, which has Euler Char $=0$, follows that $|V| - |E| + |F| = 0 \Rightarrow \boxed{\dim \mathcal{L}_\Gamma = |F|+1}$

For a Γ on another surface (or planar in a disk) we get variant formulae analogously.

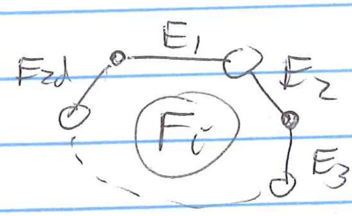
Equivalently, $\dim \mathcal{L}_\Gamma = \underbrace{(|F|-1)}_{\text{cycles determined by a face, for all but one face}} + \underbrace{2}_{\text{two homology cycles of torus.}}$

Orienting all faces clockwise, F_i determines a weight

$w(F_i) = E_1 E_2^{-1} E_3 E_4^{-1} \cdots E_{2d}^{-1}$ if



or $w(F_i) = E_1^{-1} E_2 E_3^{-1} E_4 \cdots E_{2d}$ for



Hence consistent by saying

$$\prod_{i=1}^{|F|} w(F_i) = 1$$

& that is the only relation.

③ **Remark:** $\mathcal{L}_\Gamma \cong \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{C}^*) = H^1(\Gamma, \mathbb{C}^*)$

$H_1(\Gamma, \mathbb{Z}) \underset{\text{canonical isomorphism}}{\cong} H_1(\hat{\Gamma}, \mathbb{Z})$ where Γ is on surface S and $\hat{\Gamma}$ is on the conj. surf. $\hat{S}$.

We also let S_0 be the punctured surface with a puncture for each face F of graph Γ.

The Canonical Isomorphism sends loops around faces F in S_0 to loops given by zig-zag paths in $\hat{S}$ (where each of these loops have same edge labels in same cyclic order).

Let Λ_Γ be the lattice defined by homology $H_1(\Gamma, \mathbb{Z}) \cong H_1(\hat{\Gamma}, \mathbb{Z})$.

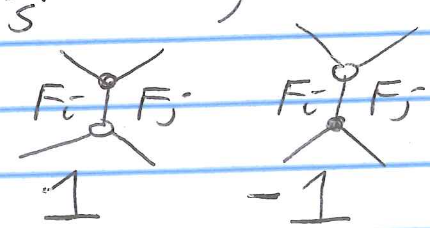
Λ_Γ has basis of cycles $\gamma_1, \ldots, \gamma_{|F|}, \alpha_1, \ldots, \alpha_{2g}$ (g = genus)
where γ_i is loop given by boundary around face F_i,
α_j is the jth fundamental cycle of unpunctured surface.

The multiplicative identity $\prod_{i=1}^{|F|} w(F_i) = 1$ becomes

the additive identity $\sum_{i=1}^{|F|} \gamma_i = 0$.

Induces an intersection pairing from $\hat{S}_\Gamma$
$\left(\gamma_i \text{ is a boundary around face } F_i \text{ on } S_0 \underset{\text{canonical}}{\cong} \text{zig-zag path on } \hat{S}\right)$

$\varepsilon_{ij} := \langle \gamma_i, \gamma_j \rangle_\Gamma =$ sum up contributions from



and extend by the Leibniz rule.

(4) Λ_Γ w/ skew-symmetric $\mathbb{Z}$-bilinear form yields a <u>quantum torus $*$-algebra</u> T_{Λ_Γ}.

T_{Λ_Γ} has a basis $\{X_v\}$ over $\mathbb{Z}[q, q^{-1}]$ parameterized by vectors v in Λ_Γ.

Multiplication given by $q^{-\langle v_1, v_2 \rangle} X_{v_1} X_{v_2} = X_{v_1+v_2}$.

$* : T_{\Lambda_\Gamma} \to T_{\Lambda_\Gamma}$ by $*(X_v) = X_v$, $*(q) = q^{-1}$.

Since $X_{v_1+v_2} = q^{-\langle v_2, v_1 \rangle} X_{v_2} X_{v_1}$ also

$\Rightarrow \boxed{X_{v_2} X_{v_1} = q^{-2\langle v_1, v_2 \rangle} X_{v_1} X_{v_2}}$ since $\langle \cdot, \cdot \rangle$ is skew-symmetric

Iterating, $X_v = q^{-\sum_{i<j} a_i a_j \langle v_i, v_j \rangle} \prod_{i=1}^{n} X_i^{a_i}$ (as an ordered product)

if $v = \sum_{i=1}^{n} a_i v_i$ w/ $a_i \in \mathbb{Z}$ and

$\{v_1, \dots, v_n\}$ is a basis for Λ_Γ, w/ total order.

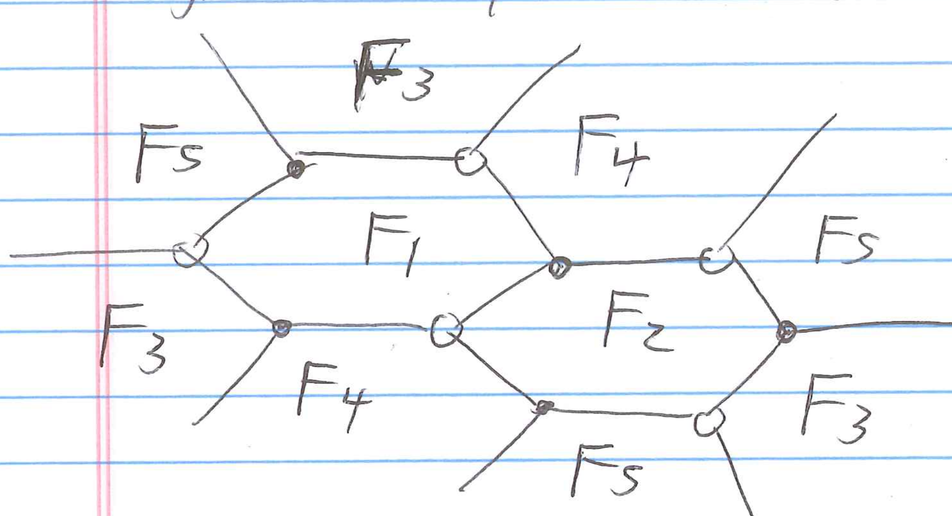
In the above, we included the non-commutative quantum structure to motivate the Poisson structure.

But we let <u>$q=1$</u> To get a commutative <u>torus algebra</u> but still with a Poisson structure.

(5) Poisson structure $\{X_i, X_j\} = \varepsilon_{ij} X_i X_j$

we similarly define $\{X_i, Z_j\}$ and $\{Z_i, Z_j\}$ for the fundamental cycles of surface S.

e.g, from Kenyon's slides (genus 1 example)



$|F| = 5$
$|E| = 15$
$|V| = 10$

$$\begin{array}{c|c} \begin{matrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ z_1 \\ z_2 \end{matrix} & \left[\begin{array}{ccccc|cc} 0 & 1 & 2 & -2 & -1 & 1 & -2 \\ -1 & 0 & 1 & 2 & -2 & 0 & 0 \\ -2 & -1 & 0 & 1 & 2 & -2 & 4 \\ 2 & -2 & -1 & 0 & 1 & 0 & 0 \\ 1 & 2 & -2 & -1 & 0 & 1 & -2 \\ \hline -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 2 & 0 & -4 & 0 & 2 & 0 & 0 \end{array}\right] \end{array} = \varepsilon_{ij}$$

in $X_1, \dots, X_{|F|-1}$

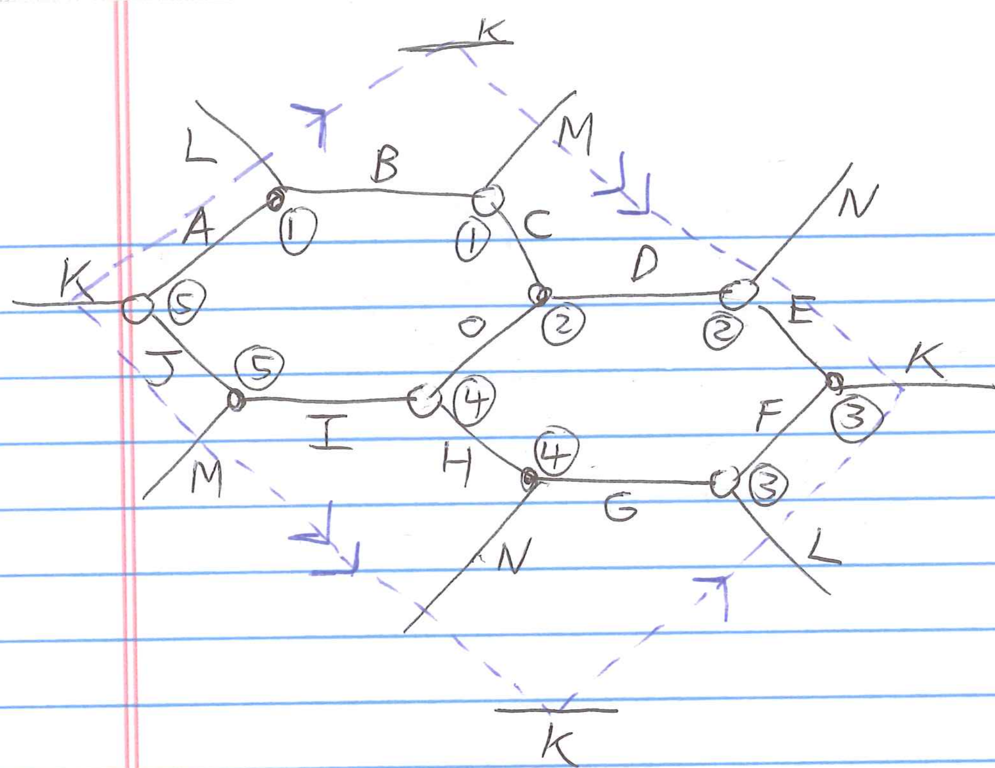
Recall the Leibniz rule $\boxed{\{AB, C\} = A\{B,C\} + \{A,C\}B}$.

$\Rightarrow \{x^n, y\} = n\{x,y\}x^{n-1}$ and $\{\frac{1}{x}, y\} = \frac{-1}{x^2}\{x,y\}$.

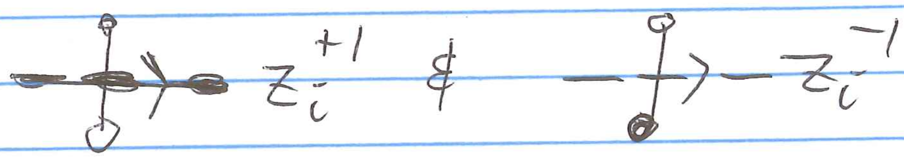
Hamiltonians $H_1, \dots, H_m$ are functions in the X_i's & Z_j's such that $\{H_i, H_j\} = 0$ for all $i, j = 1, \dots, m$. Called commuting or in involution.

Casimirs C_i commute with everything, meaning $\{C_i, X_j\} = \{C_i, Z_j\} = 0$.

⑥



We define the (z_1, z_2)-weighted Kastelyn matrix as the weighted adjacency matrix between Black & White vertices. Define $z_1^a z_2^b$ – weighting of an edge by multiplying together contributions



For each fundamental cycle α_i of the torus.

Example: $K = \begin{bmatrix} B & \emptyset & L z_1^{-1} & \emptyset & A \\ C & D & \emptyset & o & \emptyset \\ \emptyset & E & F & \emptyset & K z_1 z_2^{-1} \\ \emptyset & N z_2 & G & H & \emptyset \\ M z_2 & \emptyset & \emptyset & I & J \end{bmatrix}$

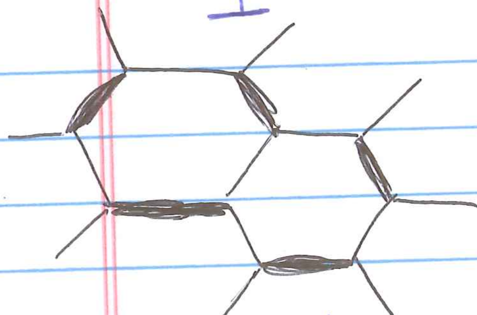
in this case.

Note: For non-hexagonal lattice, also need certain signs on entries of Kastelyn Matrix K.

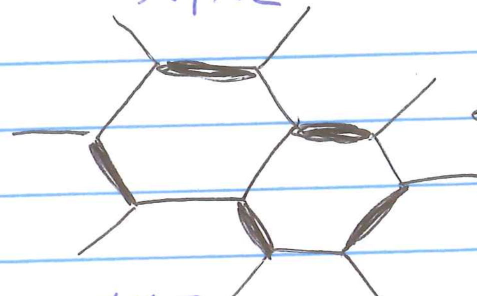
⑦ Summands of $\det K$ correspond to perfect matchings, i.e. dimer covers

E.g.

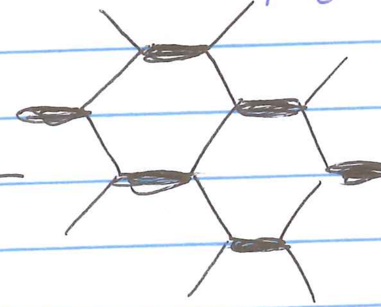
1



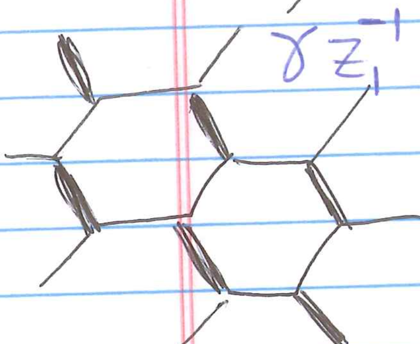
$x_1 x_2$



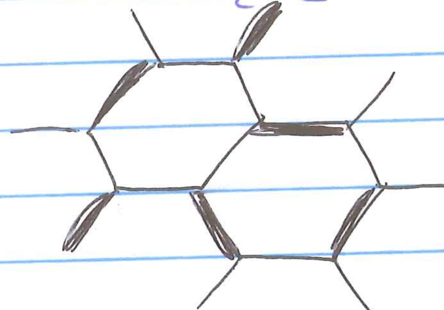
$\alpha^{-1}\gamma\delta z_1 z_2^{-1}$



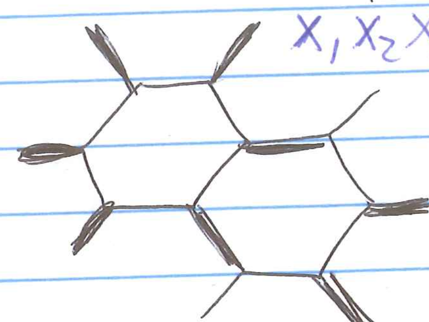
γz_1^{-1}



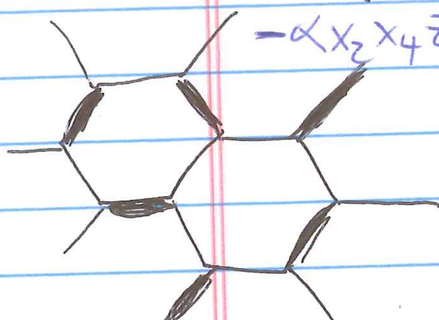
$-\alpha x_2 z_2$



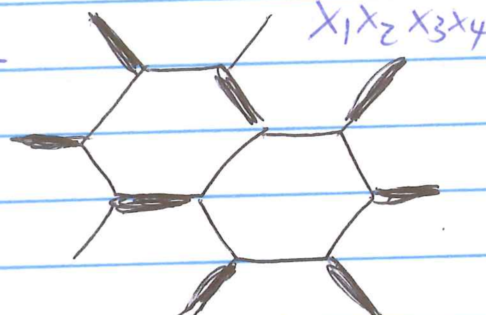
$x_1 x_2 x_3$



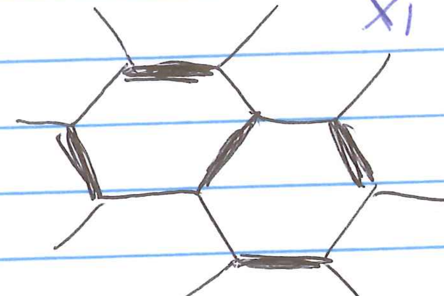
$-\alpha x_2 x_4 z_2$



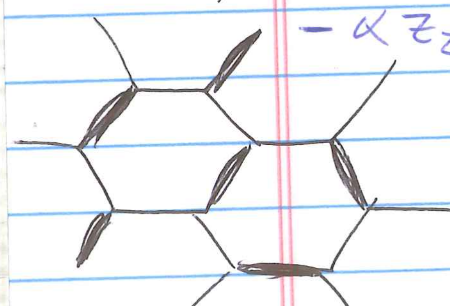
$x_1 x_2 x_3 x_4$



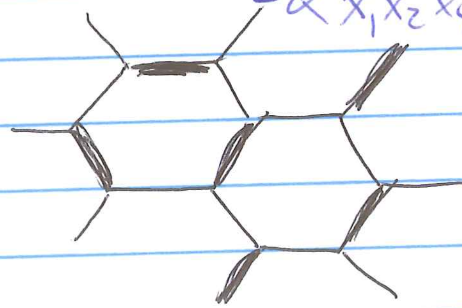
x_1



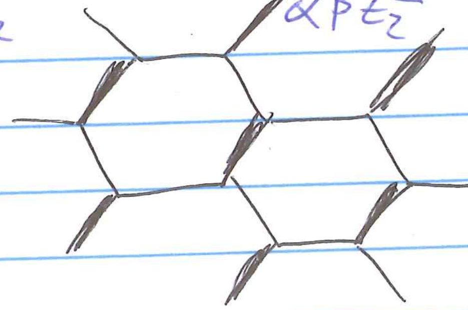
$-\alpha z_2$



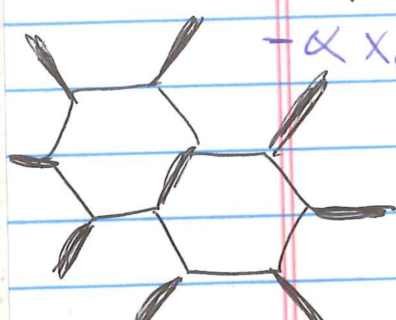
$-\alpha x_1 x_2 x_4 z_2$



$\alpha\beta z_2^2$



$-\alpha x_1 x_2 x_3 x_4 z_2$



$\det K$ has summands

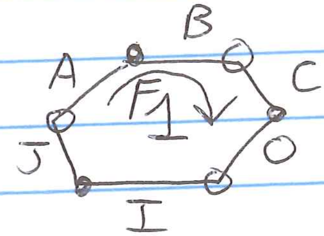
ACEGI, BDFHJ, BDGIK,
CEHJL, ADFHM, DHKLM,
ACFIN, CIKLN, BEGJO,
AEGMO, BFJNO, AFMNO,
KLMNO (Matching the above order)

(8) In Blue, the weights assigned to each perfect matching are

- sign & $z_1^a z_2^b$-weight as in $\det K$
- W.l.o.g. one of these matchings w/ $z_1^0 z_2^0$-weight is called "1"

Then all other matchings also w/ $z_1^0 z_2^0$-weight is a product of X_i's where

$X_1 = \dfrac{BJO}{ACI}$ corresponding to the face weight



e.g. $BEGJO = ACEGI \cdot X_1$

Similarly, $X_2 = \dfrac{DFH}{EGO}$, $X_3 = \dfrac{KLM}{BFJ}$, $X_4 = \dfrac{CIN}{DHM}$

Note that $X_5 = \dfrac{AEG}{KLN}$ & $X_1 X_2 X_3 X_4 X_5 = 1$.

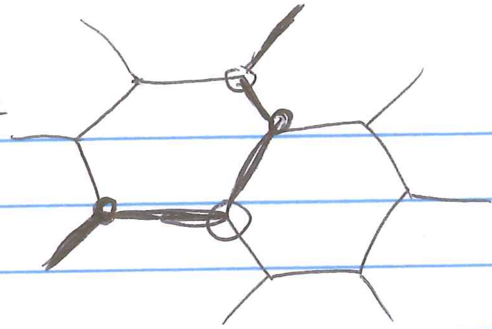
We let $\alpha, \beta, \gamma, \delta, z_1, z_2$ be defined in such a way s.t.

$\alpha z_2 = \dfrac{MO}{CI}$, $\alpha \beta z_2^2 = \dfrac{MO}{CI} \cdot \dfrac{FN}{EG} \Rightarrow \beta = \alpha X_2 X_4$,

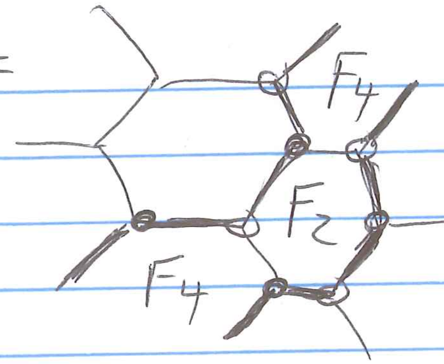
$\gamma z_1^{-1} = \dfrac{HJL}{AGI}$, $\alpha^{-1} \gamma \delta z_1 z_2^{-1} = \dfrac{BDK}{ACE}$

(9)

$\alpha z_2 =$

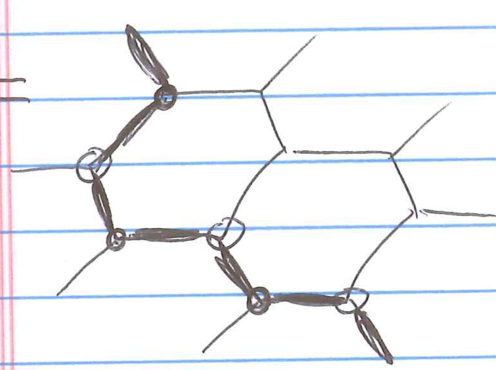


$\alpha \beta z_2^2 =$

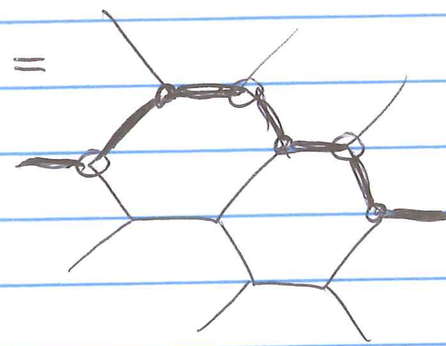


Notice $\beta = \alpha x_2 x_4$

$\gamma z_1^{-1} =$



$\alpha^{-1} \gamma \delta z_1 z_2^{-1} =$



Note:

Conjugate surface $\hat{S}$ in this example has faces

A B M I O D N G F K

A J M C O H N E F L

B L G H I J K E D C

[correspond to zig-zag paths on S]

$|V| = 10, \quad |E| = 15, \quad |\hat{F}| = 3 \Rightarrow \boxed{\text{genus}(\hat{S}) = 2}$

$\chi = 10 - 15 + 3 = -2 = 2 - 2g$

Even though genus $(S) = 1$. (w/ $|F| = 5$)

But we still have $H_1(\Gamma, \mathbb{Z}) \cong H_1(\hat{\Gamma}, \mathbb{Z})$

since homology of the graphs Γ, resp. $\hat{\Gamma}$ have cycle basis of BOTH

loops around faces + fund. cycles of surface

In this e.g.

$H_1(\Gamma, \mathbb{Z})$ has basis $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \alpha_1, \alpha_2$

but $H_1(\hat{\Gamma}, \mathbb{Z})$ " " $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$

so still are canonically isomorphic.