11/12/18

From the Octahedron Recurrence to Gale-Robinson Sequences and other discrete integrable systems

Given the dual quiver for the $\mathbb{Z}^2$-checkerboard lattice

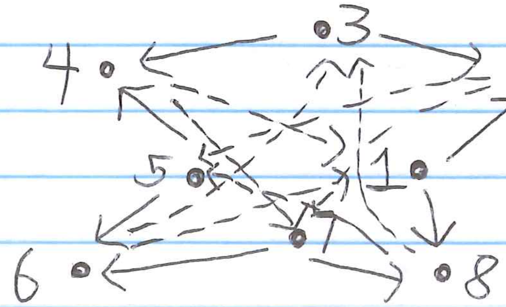
$$\begin{array}{ccccc}
 & & \vdots & & \\
-11 & \rightarrow & 01 & \leftarrow & 11 \\
\uparrow & & \downarrow & & \uparrow \\
\cdots\ -10 & \leftarrow & 00 & \rightarrow & 10\ \cdots \\
\downarrow & & \uparrow & & \downarrow \\
-1-1 & \rightarrow & 0-1 & \leftarrow & 1-1 \\
 & & \vdots & &
\end{array}$$

we can covert this into a $\mathbb{Z}$-labeled quiver by sending $(r,s) \longmapsto \underline{r+3s}$

$$\begin{array}{ccccc}
 & & \vdots & & \\
4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 \\
\cdots\ -2 & -1 & 0 & 1 & 2\ \cdots \\
-5 & -4 & -3 & -2 & -1 \\
-8 & -7 & -6 & -5 & -4 \\
 & & \vdots & &
\end{array}
\qquad \mathbb{Z}^2 / \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

For a given pentagram quiver Q_n, we can identify values modulo $2n$ & take the resulting projection

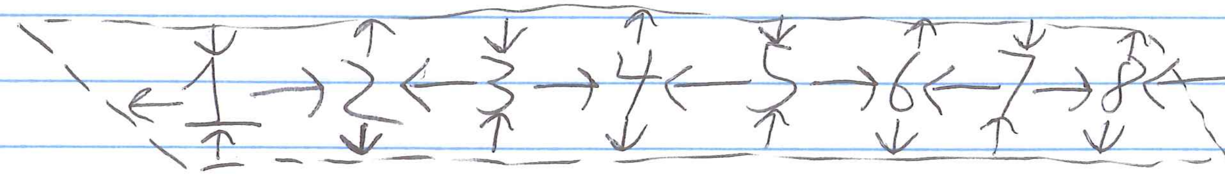
E.g. Q_4



$$\begin{array}{ccccccccccccccccc}
4 & \leftarrow & 5 & \rightarrow & 6 & \leftarrow & 7 & \rightarrow & 8 & \leftarrow & 1 & \rightarrow & 2 & \leftarrow & 3 & \rightarrow & 4 \\
\downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow \\
1 & \rightarrow & 2 & \leftarrow & 3 & \rightarrow & 4 & \leftarrow & 5 & \rightarrow & 6 & \leftarrow & 7 & \rightarrow & 8 & \leftarrow & 1 \\
\uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow \\
6 & \leftarrow & 7 & \rightarrow & 8 & \leftarrow & 1 & \rightarrow & 2 & \leftarrow & 3 & \rightarrow & 4 & \leftarrow & 5 & \rightarrow & 6
\end{array}$$

$\mathbb{Z}^2 / \langle \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2n \\ 0 \end{bmatrix} \rangle$

(2) These are examples of quivers on tori.

$Q_4 \cong$ 

M. Glick's formula for Y-system elements/F-polys involve perfect matchings of Aztec Diamonds which are subgraphs of the $\mathbb{Z}^2$-lattice we started with.

In the $\mathbb{Z}^2$-dual quiver, mutation sequences lead to octahedron recurrence

$$X_{i,j,k+1} X_{i,j,k-1} = X_{i-1,j,k} X_{i+1,j,k} + X_{i,j-1,k} X_{i,j+1,k}$$

projecting down to Q_n on $\mathbb{Z}^2 / \langle \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2n \\ 0 \end{bmatrix} \rangle$ also projects the Aztec Diamond combinatorial interpretation

We now switch gears and consider other quivers on tori, and how combinatorics of Aztec Diamond-like graphs relate in these other cases.

Recall the Somos-4, Somos-5 sequences:

$$x_n x_{n-4} = x_{n-1} x_{n-3} + x_{n-2}^2,$$

$$x_n x_{n-5} = x_{n-1} x_{n-4} + x_{n-2} x_{n-3}.$$

(3) These are both examples of sequences from 1-periodic quivers.

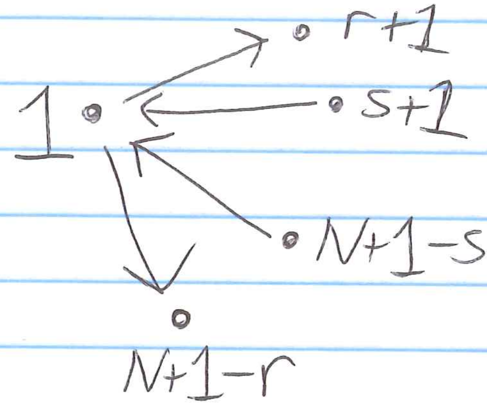
Part of larger family: <u>Gale-Robinson sequences</u>

$$x_n x_{n-N} = x_{n-r} x_{n-N+r} + x_{n-s} x_{n-N+s} \quad \text{for } n > N$$

assuming $1 \leq r < s \leq \lfloor \frac{N}{2} \rfloor$

Somos-4 $\longleftrightarrow$ $r=1, s=2, N=4$
Somos-5 $\longleftrightarrow$ $r=1, s=2, N=5$

Locally



around vertex 1.

Since palindromic symmetry, there is a unique 1-periodic quiver which can be completed from this vertex-1-neighborhood.

$$\approx P_N^{(r)} - P_N^{(s)} + \sum \varepsilon_{ij}\text{-subquivers}$$

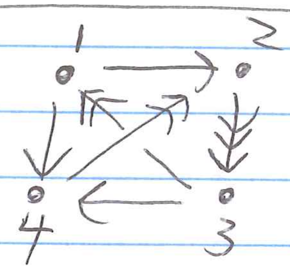
[In special case $s = N/2$ & N even, $\approx P_N^{(r)} - 2P_N^{(s)} + \sum \varepsilon_{ij}\text{-subq}$]

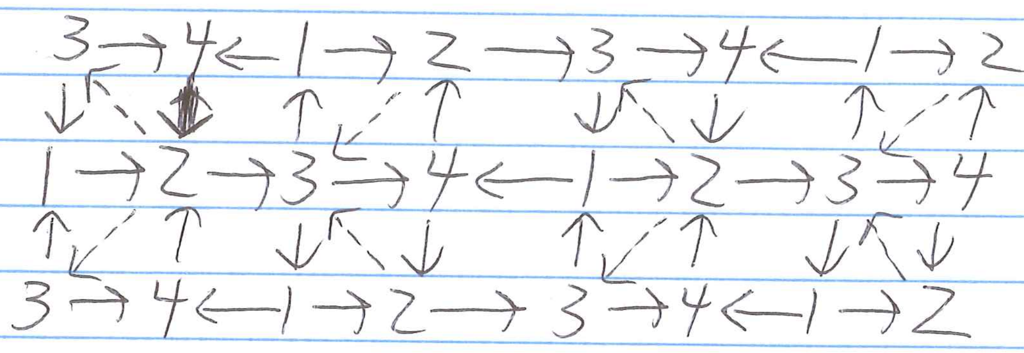
Mutating (r,s,N)-Gale-Robinson quiver periodically
$1, 2, 3, \ldots, N, 1, 2, 3, \ldots, N$
yields cluster variables $\longleftrightarrow$ Gale-Robinson sequence.

(4) Like the Glick Pentagram quivers, these quivers may also be <u>drawn on a torus</u>.

E.g. Somos-4 quiver $(r=1, s=2, N=4)$

$$P_4^{(1)} - 2P_4^{(2)} + 2P^{(1)}_{\{2,3\}}$$





<u>Step 1</u>: Skip by <u>r</u> along horizontals (to the right)
orient arrows to <u>larger number</u> (work <u>mod N</u>)

<u>Step 2</u>: Skip by <u>s</u> along verticals (up)
orient arrows to <u>smaller number</u> (work <u>mod N</u>)

<u>Step 3</u>: Complete w/ <u>diagonals</u> so all faces are cyclic triangles or squares.

In this e.g. <u>two types of diagonals</u>, both $3 \leftarrow 2$.
In general, correspond to the ε_{ij}-subquivers in 1-periodic quiver construction.