From the Octahedron Recurrence to Gale-Robinson Sequences and other discrete integrable systems

Given the dual quiver for the $\mathbb{Z}^2$-checkerboard lattice

```
-11 → 01 ← 11
↑    ↓    ↑
(...)
-10 ← 00 → 10 (...)
↓    ↑    ↓
-1-1 → 0-1 ← 1-1
```

we can convert this into a $\mathbb{Z}^2$-labeled quiver by sending $(r,s) \mapsto \mathbb{P} + 3s$

```
4 5 6 7 8
1 2 3 4 5
(...)
-2 -1 0 1 2 (...)
-5 -4 -3 -2 -1
-8 -7 -6 -5 -4
```

For a given pentagram quiver $Q_n$, we can identify values modulo $\mathbb{Z}_n$ and take the resulting projection

E.g., $Q_4$

```
\begin{array}{cccc}
4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 \\
(...)
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 1 & 2 & 3 & 4
\end{array}
```

```
4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4
↓ ↑ ↓ ↑ ↓ ↑ ↓ ↑
1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 1
↑ ↓ ↑ ↓ ↑ ↓ ↑ ↑
6 \rightarrow 7 \rightarrow 8 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6
```
These are examples of quivers on tori.

\[ \mathbb{Z}^2 / \langle [-3], [z_0] \rangle \]

\[ q_4 = \]

M. Glick's formula for Y-system elements/F-polys involve perfect matchings of Aztec Diamonds which are subgraphs of the \( \mathbb{Z}^2 \)-lattice we started with.

In the \( \mathbb{Z}^2 \)-dual quiver mutation sequences lead to Octahedron recurrence

\[
\frac{x_{ij,k+1}}{x_{ij,k-1}} = \frac{x_{i-1,j,k} + x_{i+1,j,k} + x_{i,j+1,k} + x_{i,j-1,k}}{x_{ij,k} + x_{i,j,k+1} + x_{i,j,k-1}}
\]

Projecting down to \( \mathbb{Q} \) on \( \mathbb{Z}^2 / \langle [-3], [z_0] \rangle \) also projects the Aztec Diamond combinatorial interpretation.

We now switch gears and consider other quivers on tori and how combinatorics of Aztec Diamond -like graphs relate in these other cases.

Recall the Somos-4, Somus-5 sequences:

\[
x_n x_{n-4} = x_{n-1} x_{n-3} + x_{n-2}^2
\]

\[
x_n x_{n-5} = x_{n-1} x_{n-4} + x_{n-2} x_{n-3}
\]
These are both examples of $1$-periodic quivers. Part of larger family: Gale-Robinson sequences

\[ X_n X_{n-N} = X_{n-r} X_{n-N+r} + X_{n-s} X_{n-N+s} \quad \text{for } n > N \]

assuming $1 \leq r < s \leq \left\lfloor \frac{N}{2} \right\rfloor$

Somos-4 $\iff r = 1, s = 2, N = 4$

Somos-5 $\iff r = 1, s = 2, N = 5$

Locally

\[ \begin{array}{c}
\circ \quad 1 \\
\circ \quad r+1 \\
\circ \quad s+1 \\
\circ \quad N+1-s \\
\circ \quad N+1-r \\
\end{array} \]

around vertex 1.

Since palindromic symmetry there is a unique $1$-periodic quiver which can be completed from this vertex-$1$-neighborhood:

\[ \asymp p_N^{(r)} - p_N^{(s)} + \sum E_{ij} \text{-subquivers} \]

In special case $s = N/2 \neq N$ even, \[ \asymp p_N^{(r)} - 2p_N^{(s)} + \sum E_{ij} \text{-subquivers} \]

Mutating $(s, N)$-Gale-Robinson quiver periodically \[ 1, 2, 3, \ldots, N, 1, 2, 3, \ldots, N \]
yields cluster variables $\leftrightarrow$ Gale-Robinson sequence.
Like the Glick Pentagram quivers, these quivers may also be drawn on a torus.

E.g., Sonos - 4 quiver \((r=1, s=2, N=4)\)

\[
P_4^{(1)} - 2P_4^{(2)} + 2P_4^{(1)} \quad \begin{array}{c}
\circ \rightarrow \circ \\
\circ \downarrow \circ \uparrow \circ \uparrow \circ \uparrow \\
\circ \uparrow \circ \uparrow \circ \downarrow \circ \downarrow \\
\circ \downarrow \circ \downarrow \circ \uparrow \circ \uparrow \\
\circ \uparrow \circ \uparrow \circ \downarrow \circ \downarrow \\
\circ \downarrow \circ \downarrow \circ \uparrow \circ \uparrow \\
\end{array}
\]

\[3 \rightarrow 4 \leftarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 1 \rightarrow 2\]

Step 1: Skip by \(r\) along horizontal (to the right)

orient arrows to larger number (work mod \(N\))

Step 2: Skip by \(s\) along verticals (up)

orient arrows to smaller number (work mod \(N\))

Step 3: Complete with diagonals so all faces are cyclic triangles or squares.

In this e.g. two types of diagonals, both \(3 \leftarrow 2\).

In general, correspond to the e.g. subquivers in 1-periodic quiver construction.