11/5/18

The Combinatorics of Aztec Diamonds and the Octahedron Recurrence

Last week we saw two examples of dynamical systems and their relation to cluster algebras.

Firstly we had the Somos-4 & Somos-5 sequences

e.g. $x_n x_{n-4} = x_{n-1} x_{n-3} = x_{n-2}^2$

$x_n x_{n-5} = x_{n-1} x_{n-4} = x_{n-2} x_{n-3}$,

discrete dynamical systems whose ensuing Laurent Polys can be interpreted as cluster variables for a periodic quiver.

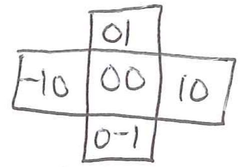
Secondly, we had the geometric pentagram map, which could be coordinized by y-parameters to yield a Y-system for a certain family of bipartite quivers. Further, the resulting Y-system was made up of F-polynomials that we related to perfect matchings of Aztec Diamonds.

Today, we study Aztec Diamonds and their perfect matchings in more depth, proving Friday's assertions and a more general family of quivers whose mutations give rise to specializations of the Octahedron Recurrence.

Def: Consider the infinite checkerboard or $\mathbb{Z}^2$-lattice with faces labeled as (i,j). Let $AD_K(i,j)$ denote the Aztec Diamond of width & height $2K-1$ centered at (i,j). It is the subgraph induced by the set of faces

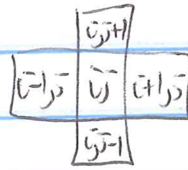
$$\{(a,b) \in \mathbb{Z}^2 \text{ s.t. } |a-i| + |b-j| \leq K-1\}.$$

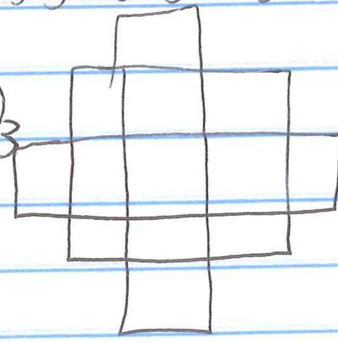
②



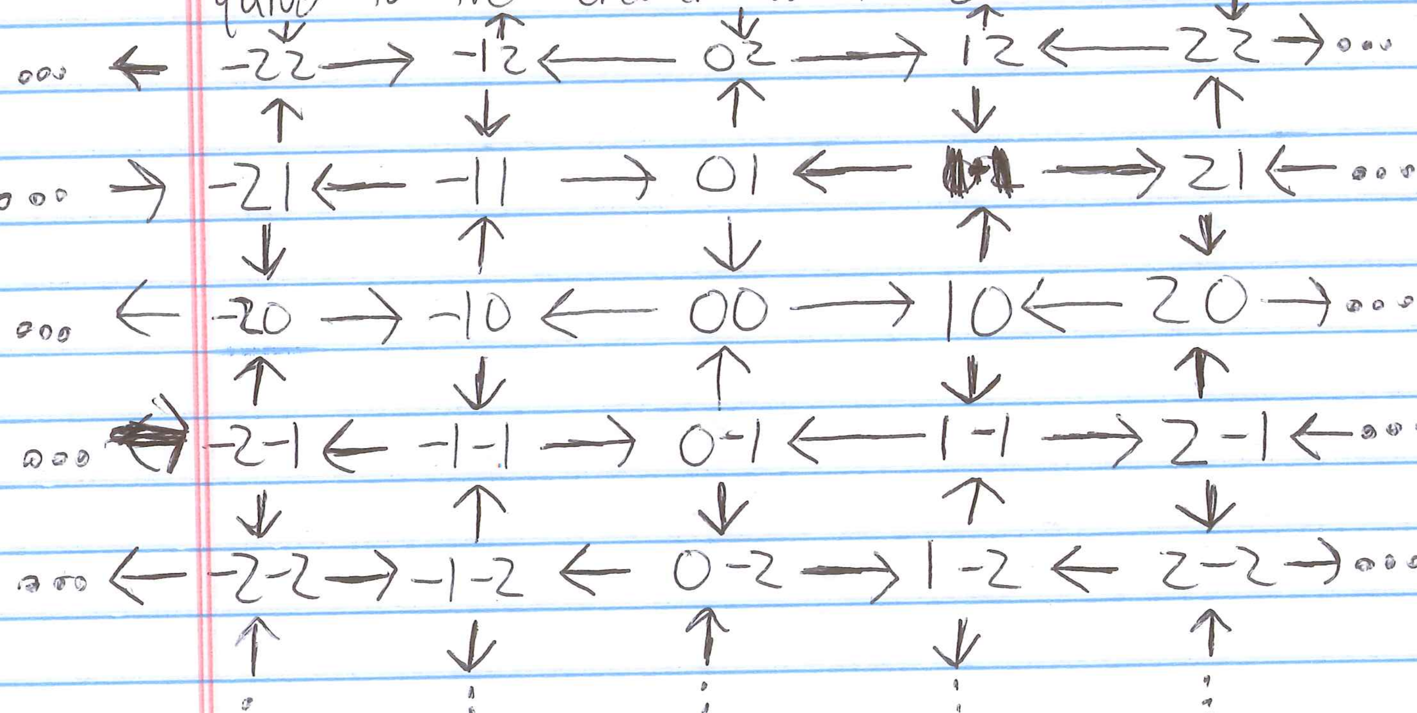
E.g., $AD_2(0,0)$ has faces $\{(-1,0),(0,-1),(0,0),(0,1),(1,0)\}$

$AD_1(i,j)$, $AD_2(i,j)$, AD_3 ...





We associate an infinite (but locally finite & doubly periodic) quiver to the checkerboard or $\mathbb{Z}^2$ lattice:



Dual Quiver to the Checkerboard lattice

Notice, the vertices (i,j) come in 4 possible local configurations depending on $i \bmod 2$ and $j \bmod 2$.

Let $\mu_{even,even}$ = Mutate at all (i,j) w/ $i \equiv 0 \bmod 2$, $j \equiv 0 \bmod 2$

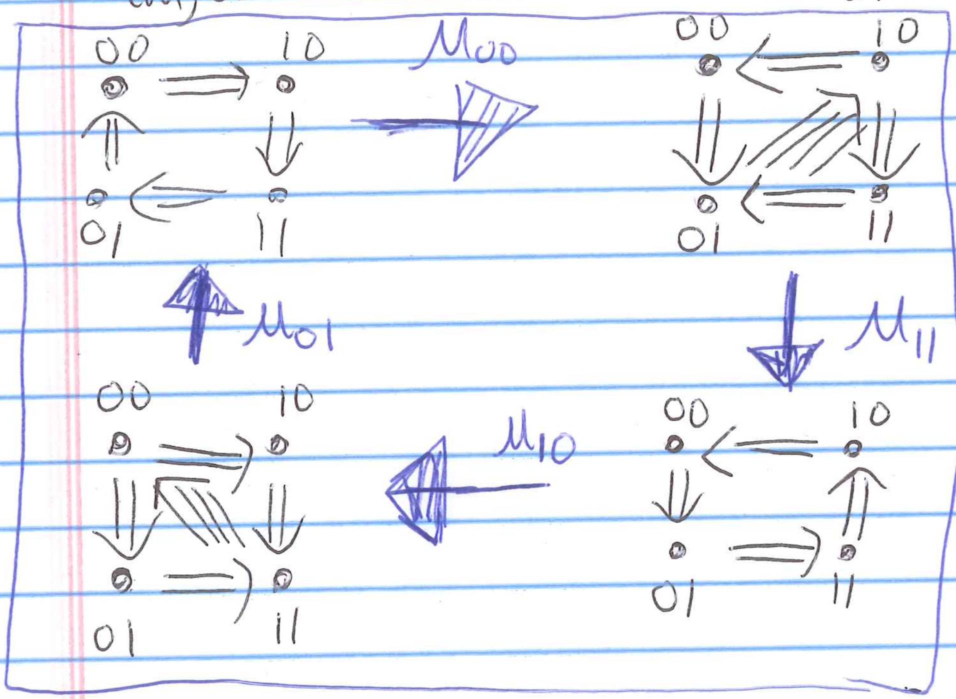
Similarly we define $\mu_{even,odd}$, $\mu_{odd,odd}$, and $\mu_{odd,even}$.

③ We can project this infinite quiver down to a 4-vertex quiver

$$\begin{tikzcd} 00 \arrow[r, Rightarrow] & 10 \arrow[d, Rightarrow] \\ 01 \arrow[u, Rightarrow] & 11 \arrow[l, Rightarrow] \end{tikzcd}$$

taking values mod $\mathbb{Z}_2$.

$\mu_{even,even}$ has same effect as μ_{00} in 4-vertex quiver.
$\mu_{odd,odd}$ " " μ_{11} " "
$\mu_{odd,even}$ " " μ_{10} " "
$\mu_{even,odd}$ " " μ_{01} " "



Thus we may iterate $\underline{\mu_{even,even}}$; $\underline{\mu_{odd,odd}}$; $\underline{\mu_{odd,even}}$; $\underline{\mu_{even,odd}}$ on the infinite quiver in that order and return back to the original.

In fact this is limit $m \to \infty, n \to \infty$ of $\underline{A_n \times A_m}$ quiver we studied Zamolodchikov periodicity of.

(4) Let x_{ij} be the initial cluster variable assoc. to $(i,j) \in \mathbb{Z}^2$.

After these <u>four composite mutations</u>, every vertex has mutated precisely once and

$$(*) \quad x_{ij}' \, x_{ij} = x_{i-1,j} \, x_{i+1,j} + x_{i,j-1} \, x_{i,j+1}.$$

<u>To index this better,</u> let the initial cluster be

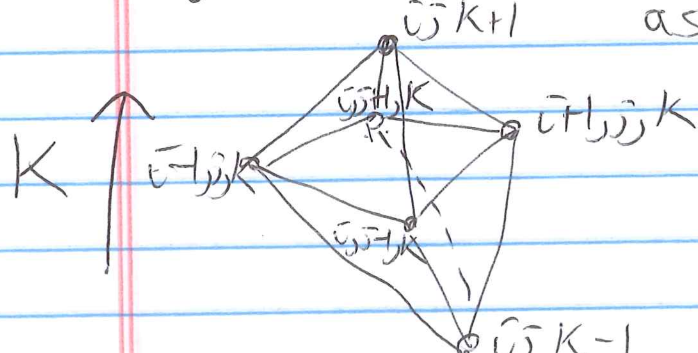
$$\{\cancel{x}_{i,j,0} : i+j \equiv 0 \bmod 2\} \cup \{x_{i,j,1} : i+j \equiv 1 \bmod 2\}$$

i.e. $\left\{ x_{ijk} : \begin{array}{l} i+j+k \equiv 0 \bmod 2 \\ k \in \{0,1\} \end{array} \right\}$ [Some references let $i+j+k \equiv 1$ instead]

We iterate to get a new cluster involving x_{ijk}'s w/ $k \geq 2$ by $\boxed{x_{ijk}' := x_{i,j,k+2}}$. Hence we can rewrite $(*)$ as

$$(**) \quad \boxed{x_{i,j,k+1} \, x_{i,j,k-1} = x_{i-1,j,k} \, x_{i+1,j,k} + x_{i,j-1,k} \, x_{i,j+1,k}}$$

Eqn $(**)$ called the Octahedron Recurrence as illustrated in $\mathbb{Z}^3$.



Can also <u>run recurrence backwards to get</u> x_{ijk} for $k < 0$.

5 Thm (Speyer '08 as part of a larger result)

If $i+j+k \equiv 0 \mod 2$, the Laurent expansion of x_{ijk} has a combinatorial interpretation as

$$x_{ijk} = \sum_{\substack{M \text{ is a perfect} \\ \text{matching of } AD_k(i,j).}} \text{Weight}(M)$$

where weight(M) is a Laurent monomial determined by perfect matching M and the face-weightings of $AD_k(i,j)$ & $AD_{k+1}(i,j)$.

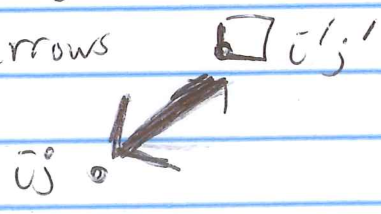
Example: $x_{ij2} = \dfrac{x_{i-1,j,1}\, x_{i+1,j,1} + x_{i,j-1,1}\, x_{i,j+1,1}}{x_{ij0}}$

if $i+j \equiv 0 \mod 2$.
(even & even OR odd & odd)

This specific example also proven in Kuo's Applications of Graphical Condensation from 2003, 2006.

We will prove a variant of the above w/ principal coeffs:

Endow the original infinite dual quiver to checkerboard lattice w/ another plane of vertices with arrows



for all $(i,j) \in \mathbb{Z}^2$.

Use Y_{ij}'s at these frozen vertices.

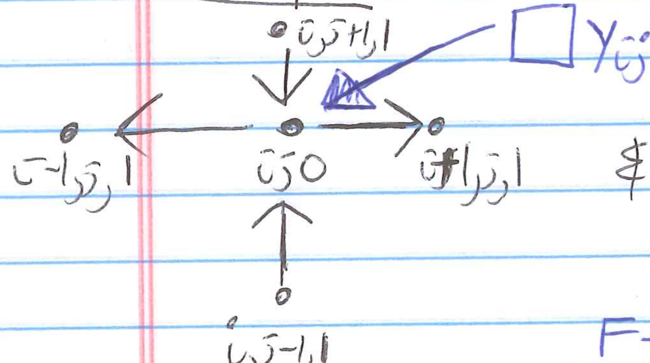
(6) <u>Claim</u>: If we keep track of principal coeffs, for <u>$i+j+k \equiv 0 \mod 2$</u>, then the Laurent expansion of x_{ijk}, in terms of the $\{x_{ij0}\}_{i+j\equiv 0 \mod 2}$ $\{x_{ij1}\}_{i+j\equiv 1 \mod 2}$, $\{y_{ij}\}$ can be written as

$$\boxed{x_{ijk} = \sum_{\substack{M \text{ is a perfect} \\ \text{matching of } AD_k(i,j)}} \text{weight}(M)\,\text{height}(M)}$$

weight(M) is a Laurent monomial in the $\{x_{ij0} : i+j \text{ even}\}$ and $\{x_{ij1} ; i+j \text{ odd}\}$

<u>height (M)</u> is an <u>ordinary monomial</u> in the y_{ij}'s.

<u>Example</u>: If $i+j \equiv 0 \mod 2$, locally have in the quiver



&

$$x_{ij2} = \frac{x_{i-1,j,1}\,x_{i+1,j,1} + \underline{y_{ij}}\,x_{i,j-1,1}\,x_{i,j+1,1}}{x_{ij0}}$$

<u>F-polynomial</u>

$1 + y_{ij}$