11/5/18

The Combinatorics of Aztec Diamonds and the Octahedron Recurrence

Last week we saw two examples of dynamical systems and their relation to cluster algebras.

Firstly we had the Somos-4 & Somos-5 sequences

\[ x_n x_{n-4} = x_{n-1} x_{n-3} = x_{n-2}^2 \]
\[ x_n x_{n-5} = x_{n-1} x_{n-4} = x_{n-2} x_{n-3} \]

Discrete dynamical systems whose ensuing Laurent Polys can be interpreted as cluster variables for a periodic quiver.

Secondly we had the geometric pentagram map which could be coordinizd by \( y \)-parameters to yield a \( Y \)-system for a certain family of bipartite quivers. Further, the resulting \( Y \)-system was made up of \( F \)-polynomials that we related to perfect matchings of Aztec Diamonds.

Today, we study Aztec Diamonds and their perfect matchings in more depth, proving Friday's assertions and a more general family of quivers whose mutations give rise to specializations of the Octahedron Recurrence.

Define consider the infinite checkerboard or \( \mathbb{Z}^2 \)-lattice with faces labeled as \((i,j)\). Let \( \text{AD}_K(i,j) \) denote the Aztec Diamond of width & height \( 2K-1 \) centered at \((i,j)\).

It is the subgraph induced by the set of faces

\[ \{ (a,b) \in \mathbb{Z}^2 \text{ s.t. } |a-i| + |b-j| \leq K-1 \} \]
E.g. \( AD_2(0,0) \) has faces \( \{(-1,0), (0,-1), (0,0), (0,1), (1,0)\} \)

\[ AD_1(i,j), \quad AD_2(i,j), \quad AD_3(i,j) \]

We associate an infinite (but locally finite & doubly periodic) quiver to the checkerboard or \( Z^2 \) lattice:

\[ \cdots \leftarrow z \rightarrow -z \leftarrow \cdots \]

\[ \cdots \rightarrow -z \leftarrow -z \rightarrow \cdots \]

\[ \cdots \leftarrow -z \rightarrow -z \leftarrow \cdots \]

\[ \cdots \rightarrow -z \leftarrow -z \rightarrow \cdots \]

\[ \text{Dual Quiver to the Checkerboard Lattice} \]

Notice the vertices \((i, j)\) come in 4 possible local configurations depending on \( i \mod 2 \) and \( j \mod 2 \).

Let \( M_{\text{even even}} \) = Mutate at all \((i, j)\) with \( i \equiv 0 \mod 2 \) and \( j \equiv 0 \mod 2 \).

Similarly, \( M_{\text{odd odd}} \) \( M_{\text{odd even}} \) and \( M_{\text{even odd}} \).
We can project this infinite quiver down to a 4-vertex quiver taking values mod 2.

Even, even has same effect as $M_{00}$ in 4-vertex quiver.

$M_{00}$

Even, odd

$M_{11}$

Odd, even

$M_{10}$

Odd, odd

$M_{01}$

Thus we may iterate $M_{even, even}$, $M_{odd, odd}$, $M_{odd, even}$, $M_{even, odd}$ on the infinite quiver in that order and return back to the original.

In fact this is limit $m \to \infty, n \to \infty$ of $A_n \times A_m$ quiver we studied Zamolodchikov periodicity of.
Let $X_{ij}$ be the initial cluster variable associated to $(i,j) \in \mathbb{Z}^2$.

After these four composite mutations, every vertex has mutated precisely once and

(1)

$X_{ij} \cdot X_{ij} = X_{i-1,j} \cdot X_{i+1,j} + X_{i,j-1} \cdot X_{i,j+1}$.

To index this better, let the initial cluster be

$X_{ij} \cap \{i+j = 0 \mod 2 \} \cup X_{ij} \cap \{i+j = 1 \mod 2 \}$

i.e., $X_{ijk} \cap \{i+j+k = 0 \mod 2 \}$, where $k \in \{0, 1\}$.

Some referees let $i+j+k = 1$ instead.

We iterate to get a new cluster involving $X_{ijk}$ with $k \geq 2$ by $X_{ijk}^\prime = X_{ijk}^{i+j+k+2}$. Hence we can rewrite (1) as

(2)

$X_{ijk}^{i+j+k+1} \cdot X_{ijk}^{i+j+k-1} = X_{i-1,j,j} \cdot X_{i+1,j,j} + X_{i,j-1} \cdot X_{i,j+1}$.

Eqn (2) called the Octahedron Recurrence $X_{ijk}^{i+j+k+1}$ as illustrated in $\mathbb{Z}^3$.

Can also run recurrence backwards to get $X_{ijk}$ for $k < 0$.  

$K$
Thm (Speyer '08 as part of a larger result)

If \( i+j+k \equiv 0 \mod 2 \), the Laurent expansion of \( x_{ij}^k \) has a combinatorial interpretation as

\[
x_{ij}^k = \sum \text{Weight}(M)
\]

where \( M \) is a perfect matching of \( \text{AD}_K(i, j) \),

weight \( (M) \) is a Laurent monomial determined by the perfect matching \( M \) and the face-weightings of \( \text{AD}(i) \cap \text{AD}(j) \).

Example: \( x_{ij}^2 = x_{i-1j+1} x_{i+j+1} + x_{i-1j-1} x_{i+j-1} \)

if \( i+j=0 \mod 2 \) (even \& even OR odd \& odd)

This specific example also proven in Kuo's Applications of Graphical Condensation from 2003, 2006.

We will prove a variant of the above w/ principal coef f s.

Endow the original infinite dual quiver to checkerboard lattice w/ another plane of vertices with arrows \( \square \)

For all \( (i,j) \in \mathbb{Z}^2 \)

Use \( Y_{ij} \) 's at these frozen vertices.
Claim: If we keep track of principal coeffs for \( i+j+k \equiv 0 \mod 2 \), then the Laurent expansion of terms of the \( \{X_{ij0}, X_{ij1}, Y_{ij}\} \) can be written as:

\[
X_{ijk} = \sum_{M} \text{weight}(M) \cdot \text{height}(M)
\]

Where:
- \( M \) is a perfect matching of \( AD_{K}(i, j) \)
- \( \text{weight}(M) \) is a Laurent monomial in the \( \{X_{ij0} \mid i+j \text{ even}\} \) and \( \{X_{ij1} \mid i+j \text{ odd}\} \)
- \( \text{height}(M) \) is an ordinary monomial in the \( Y_{ij} \)s.

Example: If \( i+j \equiv 0 \mod 2 \), locally have in the queue:

\[
X_{ij2} = X_{i-bj1} X_{i+lj1} + Y_{ij0} X_{i-j-1} X_{i+j+1}
\]

where \( \omega \) is an F-polynomial:

\[
1 + Y_{ij0} X_{i-j-1} X_{i+j+1}
\]