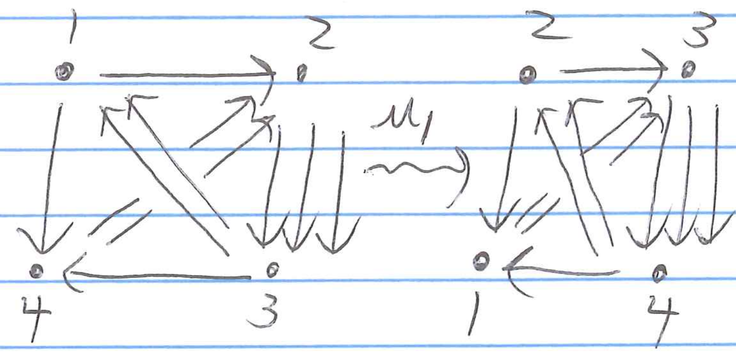
10/29/18

Def: We say that a quiver Q is <u>1-periodic in the Fordy-Marsh sense</u> if $\boxed{\mu_1(Q) = \rho Q}$ where $\rho =$ cyclic permutation $(12\cdots n)$ relabeling vertices.

<u>Example</u>: Let $Q =$



$$B = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & 0 & 3 & -2 \\ 2 & -3 & 0 & 1 \\ -1 & 2 & -1 & 0 \end{bmatrix}$$

(w/ $N = \#$ vertices)

For a <u>1-Periodic Quiver</u>, if we consider the mutation sequence $\cancel{A}\,\mu_1 \circ \mu_2 \circ \mu_3 \circ \cdots \circ \mu_N \circ \mu_1 \circ \mu_2 \circ \cdots$ repeating in order, then the resulting <u>cluster variables parametrized by $n \in \mathbb{Z}$</u>

$$x_n x_{n+N} = \prod_{i+1 \to 1} x_{n+i} + \prod_{1 \to j+1} x_{n+j} \quad \text{for } n \geq 1$$

<u>Running Example</u>: $x_n x_{n+4} = x_{n+2}^2 + x_{n+1} x_{n+3}$

Called the <u>Somos-4 sequence</u>.

If we let $x_1, x_2, x_3, x_4 = 1$, then $x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, \ldots =$

$$2, 3, 7, 23, 59, 314, 1529, \ldots$$

Always positive integers despite the division.
[Consequence of the <u>Laurent Phenomenon</u>.]

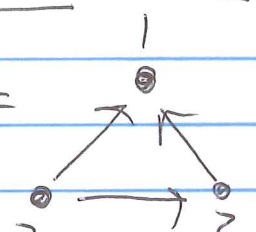
Goal: Describe all 1-periodic quivers.

We begin by describing a two-parameter family of 1-periodic Quivers that we call Primitive 1-periodic Quivers

Def: $P_N^{(K)}$ is defined as the N-vertex quiver such that for every $1 \le i \le N$, there is a single arrow joining vertex i and $i+K \pmod N$. Further we orient that arrow so that it points to the smaller index.

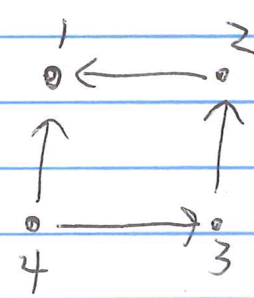
Examples: $P_2^{(1)}$

$P_3^{(1)} =$

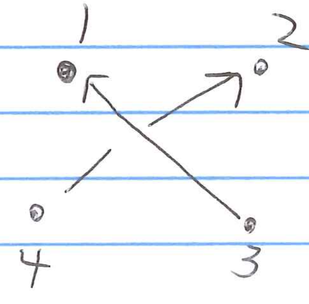


$= P_3^{(2)}$

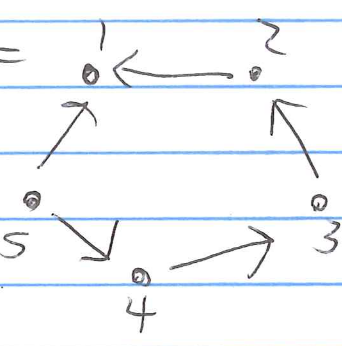
$P_4^{(1)} =$ $\overset{\|}{P_4^{(3)}}$



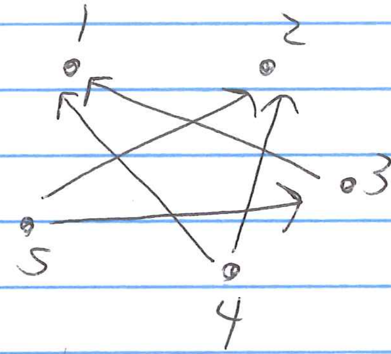
, $P_4^{(2)} =$



$P_5^{(1)} =$ $\overset{\|}{P_5^{(4)}}$



, $P_5^{(2)} =$ $\overset{\|}{P_5^{(3)}}$



W.l.o.g. we can assume $1 \le K \le \frac{N}{2}$.

Rem: $P_N^{(K)}$ connected $\iff \gcd(N,K) = 1$. If $\gcd = d$, we get d conn. components, each $\cong P_{N/d}^{(K/d)}$.

By construction, vertex 1 (the smallest entry) of $P_N^{(K)}$ is a sink, and mutation μ_1 turns 1 into a source but leaves the rest of the quiver unchanged.

Hence if relabel vertex 1 as $(N+1)$, we see

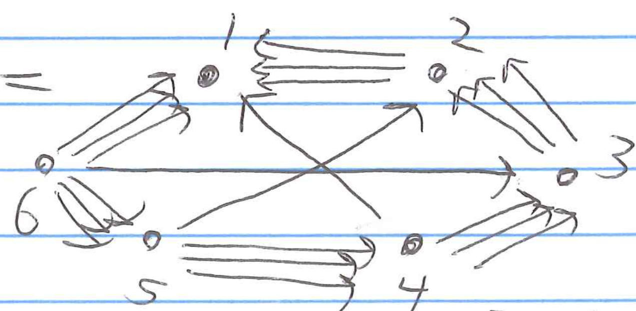
$$\mu_1(P_N^{(K)}) \equiv P^{(K)}_{\{2,3,4,\dots,N+1\}} \cong P_N^{(K)} \text{ via cyclic rotation } \rho$$

i.e. $\mu_1(P_N^{(K)}) = \rho(P_N^{(K)})$ as desired.

Claim: Any 1-periodic quiver with the further property that vertex 1 is a sink can be decomposed as a combination, possibly w/ multiplicities, of $P_N^{(i)}$'s.

We can express such a quiver as $\boxed{a_1 P_N^{(1)} + \dots + a_{\lfloor \frac{N}{2} \rfloor} P_N^{(\lfloor \frac{N}{2} \rfloor)}}$ where $a_1, \dots, a_{\lfloor \frac{N}{2} \rfloor} \in \mathbb{Z}_{\geq 0}$.

E.g. $3P_6^{(1)} + P_6^{(3)} =$

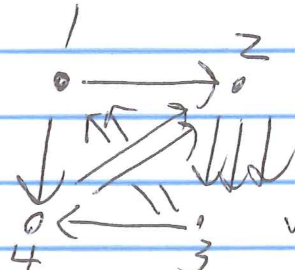


Pf: Clearly such a quiver is 1-periodic for the same reason as above: Mutation at 1 turns sink 1 into source $N+1$.

Further, if we begin w/ a 1-periodic quiver that has a_i arrows pointing from vertex $(i+1)$ to vertex 1 [we are assuming 1 is a sink] Then 1-periodicity dictates the adjacencies w/ vertex 2.

and inductively, we derive the adjacencies at the remaining vertices $\Rightarrow$ such a quiver $= c_1 P_N^{(1)} + \dots + c_K P_N^{(K)}$ where $K \leq N/2$.

The Somos-4 Quiver



is an example of a 1-periodic quiver where 1 is <u>not</u> a sink.

<u>Thm 6.6 of Fordy–Marsh</u> Any 1-periodic quiver on N vertices can be written as

$$a_1 P_N^{(1)} + a_2 P_N^{(2)} + \dots + a_{\lfloor \frac{N}{2} \rfloor} P_N^{(\lfloor \frac{N}{2} \rfloor)} + \sum_{l=1}^{\lfloor \frac{N}{2} \rfloor - 1} \sum_{t=1}^{\lfloor \frac{N}{2} \rfloor - 1} \varepsilon_{l,l+t}\, \rho^l \left(P_{N-2l}^{(t)}\right)$$

where we let $a_i \in \mathbb{Z}$ (possibly <u>negative</u>)

signifying reversal, i.e. a primitive w/ 1 a source, etc.

and ε_{ij} is a scalar defined as

$$\varepsilon_{ij} := \frac{1}{2}\left(a_i |a_j| - a_j |a_i|\right)$$

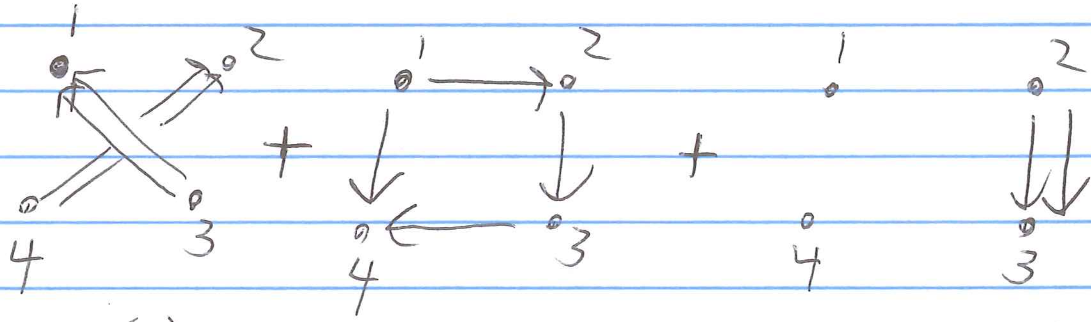
$$= \begin{cases} \operatorname{sgn}(a_j)\, a_i a_j & \text{if } a_i \text{ \& } a_j \text{ have opposite signs} \\ 0 & \text{if } \operatorname{sgn}(a_i) = \operatorname{sgn}(a_j) \end{cases}$$

<u>Special case</u>: if all $a_i \geq 0$ [like in case when vertex 1 is a sink] then $\varepsilon_{ij} = 0 \ \forall\, i,j$.

<u>Rem:</u> $\rho^l \left(P_{N-2l}^{(t)}\right)$ is a primitive on vertices $\{l+1, l+2, \dots, N-l\}$

5 **E.g.** Somos-4 Quiver $= 2P_4^{(2)} - P_4^{(1)} - 2\rho P_{4-2}^{(1)}$

$$\varepsilon_{12} = \mathrm{sgn}(+2)\cdot(-1)(2) = -2$$



$$2P_4^{(2)} \qquad -P_4^{(1)} \qquad + \qquad -2\rho P_2^{(1)}$$

Necessary condition: If B is the exchange matrix associated to a 1-periodic quiver (on N vertices) then B's first row is palindromic (after initial $b_{11}=0$)

e.g. $[b_{12}\ b_{13}\ b_{14}\ b_{15}\ b_{16}\ b_{17}] = [2\ -1\ 0\ 1\ 0\ -1\ 2]$

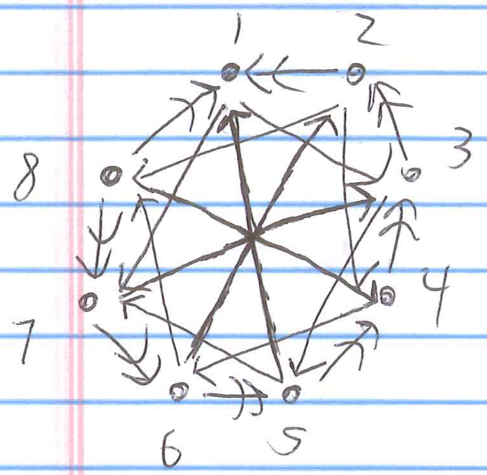
Pf: ~~B~~ By Thm 6.6, $b_{1K} = -a_{K-1}$, where a_{K-1} is the coeff of $P_N^{(K)}$ in the decomposition of the 1-periodic quiver. By def'n of $P_N^{(K)}$, $b_{1K} = -a_{K-1} = b_{1,N-K+2}$ $\Rightarrow$ Palindromicity.

As a consequence: first row of B (when palindromic as above) uniquely determines the decomposition.

above e.g. corresponds to $2P_8^{(1)} - P_8^{(2)} + P_8^{(4)} + 2\rho P_6^{(1)} - \rho^2 P_4^{(2)}$.

$2P_8^{(1)} \; 0P_8^{(2)} + 0P_8^{(3)} + P_8^{(4)} +$ terms involving ε_{ij}'s

(6) $\varepsilon_{12} = 2$, $\varepsilon_{13} = 0$, $\varepsilon_{14} = 0$, $\varepsilon_{23} = 0$, $\varepsilon_{24} = -1$, $\varepsilon_{34} = 0$



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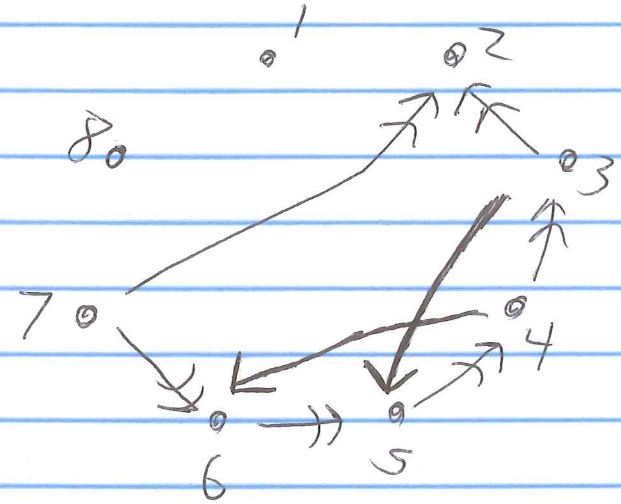


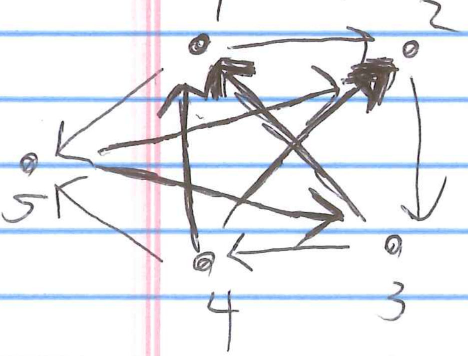
Example: Somos-5 sequence defined by

$$x_n x_{n+5} = x_{n+1} x_{n+4} + x_{n+2} x_{n+3}:$$

if $x_1 = \dots = x_5 = 1$, we get $x_6, x_7, \dots = 2, 3, 5, 11, 37, 83, 274, \dots$

defined by $\quad P_5^{(2)} - P_5^{(1)} - \rho P_{5-2}^{(1)}$

since $\varepsilon_{12} = \mathrm{sgn}(+1)\cdot(-1)\cdot(1) = 1$



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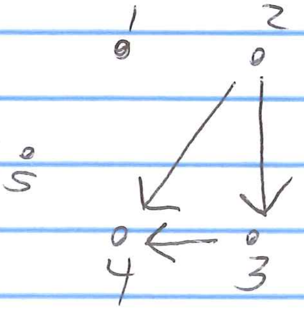

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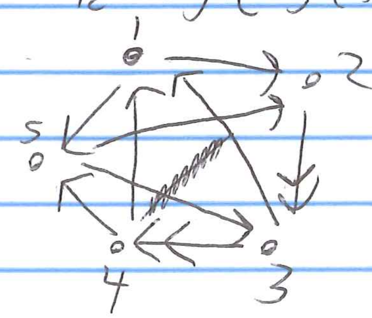

Notice 2-cycle $\boxed{2 \longleftrightarrow 4}$ deleted but double arrows $2 \twoheadrightarrow 3 \twoheadrightarrow 4$ appear.

$$x_1 x_1' = x_2 x_5 + x_3 x_4 \text{ as desired.}$$

Proof of Thm 6.6 (Classification of 1-Periodic Quivers)

Suppose Q is 1-periodic and has $N \times N$ exchange matrix B.

We wish to show $\exists\ [m_1, \dots, m_{N-1}] \in \mathbb{Z}^{N-1}$ s.t. $m_K = m_{N-K}$

and

$$(**)\quad \boxed{b_{ij} = m_{i-j} + \varepsilon_{1,i-j+1} + \varepsilon_{2,i-j+2} + \dots + \varepsilon_{j-1,i-1}} \quad \text{for } i>j.$$

$$\left(\text{recall } \varepsilon_{ij} := \tfrac{1}{2} m_i |m_j| - m_j |m_i|\right)$$

Let $\rho = \begin{bmatrix} 0 & & & 0 & 1 \\ 1 & \ddots & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{bmatrix}$, i.e. the matrix corresponding to cyclic rotation.

$\Rightarrow \rho(Q)$ associated to $\rho B \rho^{-1}$ (as an exch. matrix).

Let $|m_i| = \#$ arrows between 1 and $(i+1)$, sign determines direction.

Now suppose that we locally have $i \xrightarrow{m_{i-1}} 1 \xrightarrow{-m_{j-1}} j$ in Q.

Then we have arrows $i \longrightarrow j$ in $\mu_1 Q$.

Hence for $i>j$ & $i,j \neq 1$, we have $[\mu_1 B]_{ij} = b_{ij} + \varepsilon_{i-1,j-1}$.

To see this, note that $i \to 1$ is a difference of $(i-1)$

$1 \to j$ " " $(j-1)$

and we have m_{i-1} arrows of the first type, $-m_{j-1}$ of the second.

These signs differ to make a 2-path $\Rightarrow \varepsilon_{i-1,j-1}$ such 2-paths

(where $\varepsilon_{i-1,j-1}$ is negative for $i \longleftarrow 1 \longleftarrow j$ orientation)

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Also, $[\rho B \rho^{-1}]_{ij} = b_{i-1,j-1}$.

Thus Q 1-periodic $\Leftrightarrow$ $\boxed{b_{ij} + \varepsilon_{i-1,j-1} = b_{i-1,j-1}}$ $(*)$
for all $i>j$ w/ $i \neq 1, j \neq 1$.

By cyclic symmetry, $b_{i-j+1,1} = m_{i-j}$, thus iterating $(*)$ yields $(**)$:

In particular, $\underline{\varepsilon_{j-1,i-1} = -\varepsilon_{i-1,j-1}}$ and

$$\begin{aligned} b_{ij} &= b_{i-1,j-1} + \varepsilon_{j-1,i-1} \\ &= b_{i-2,j-2} + \varepsilon_{j-2,i-2} + \varepsilon_{j-1,i-1} \\ &\;\vdots \\ &= b_{i-j+1,1} + \varepsilon_{1,i-j+1} + \varepsilon_{2,i-j+2} + \dots + \varepsilon_{j-1,i-1} \\ &= m_{i-j} + \text{''} \qquad\qquad \text{''} = (**). \end{aligned}$$

Lastly, if we mutate at vertex N instead, we can run the whole recurrence backwards to get $m_k \leftrightarrow m_{N-k}$ symmetry. ▥