10/29/18

**Def:** We say that a quiver $Q$ is 1-periodic in the Fordy-Marsch sense if $M_1(Q) = P Q$, where $P$ = cyclic permutation (12...n) relabeling vertices.

**Example:** Let $Q = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$, $B = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & 0 & 3 & -2 \\ 2 & -3 & 0 & 1 \\ -1 & 2 & -1 & 0 \end{bmatrix}$, $N$ = # vertices

For a 1-periodic quiver, if we consider the mutation sequence $\alpha \omega_1 \omega_2 \omega_3 ... \alpha \omega_1 \omega_2 \omega_3 ...$ repeating in order, then the resulting cluster variables parametrized by $n \in \mathbb{Z}$

$$X_n X_{n+N} = \prod_{i=1}^{n} X_{n+i} + \prod_{i=1}^{n} X_{n+j} \text{ for } n \geq 1$$

**Running Example:**

$$X_n X_{n+4} = X_{n+1}^2 + X_{n+1} X_{n+3}$$

Called the Somos-4 sequence.

If we let $x_1 x_2 x_3 x_4 = 1$, then $x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} ... = 2, 3, 7, 23, 59, 319, 1529, ...$

Always positive integers despite the division. (Consequence of the Laurent phenomenon)

Goal: Describe all 1-periodic quivers.

We begin by describing a two-parameter family of 1-periodic quivers that we call \textit{Primitive 1-periodic quivers}.

Def: $P_N^{(k)}$ is defined as the $N$-vertex quiver such that for every $1 \leq i \leq N$, there is a single arrow joining vertex $i$ and $i+k \pmod{N}$. Further, we orient that arrow so that it points to the smaller index.

Examples:

$P_2^{(1)} = \begin{array}{c}
1 \\
\end{array} \xrightarrow{1} \begin{array}{c}
2 \\
\end{array}$

$P_2^{(2)} = \begin{array}{c}
1 \\
\end{array} \xleftarrow{1} \begin{array}{c}
2 \\
\end{array}$

$P_3^{(1)} = \begin{array}{c}
1 \\
\end{array} \xrightarrow{1} \begin{array}{c}
3 \\
\end{array} \xleftarrow{1} \begin{array}{c}
2 \\
\end{array}$

$P_3^{(2)} = \begin{array}{c}
1 \\
\end{array} \xrightleftharpoons{1} \begin{array}{c}
3 \\
\end{array} \xleftarrow{1} \begin{array}{c}
2 \\
\end{array}$

$P_4^{(1)} = \begin{array}{c}
1 \\
\end{array} \xrightarrow{1} \begin{array}{c}
3 \\
\end{array} \xleftarrow{1} \begin{array}{c}
2 \\
\end{array}$

$P_4^{(2)} = \begin{array}{c}
1 \\
\end{array} \xrightleftharpoons{1} \begin{array}{c}
3 \\
\end{array} \xleftarrow{1} \begin{array}{c}
2 \\
\end{array}$

$P_5^{(1)} = \begin{array}{c}
1 \\
\end{array} \xrightarrow{1} \begin{array}{c}
3 \\
\end{array} \xleftarrow{1} \begin{array}{c}
2 \\
\end{array}$

$P_5^{(2)} = \begin{array}{c}
1 \\
\end{array} \xrightleftharpoons{1} \begin{array}{c}
3 \\
\end{array} \xleftarrow{1} \begin{array}{c}
2 \\
\end{array}$

$P_5^{(3)} = \begin{array}{c}
1 \\
\end{array} \xrightleftharpoons{1} \begin{array}{c}
3 \\
\end{array} \xleftarrow{1} \begin{array}{c}
2 \\
\end{array}$

$P_5^{(4)} = \begin{array}{c}
1 \\
\end{array} \xrightleftharpoons{1} \begin{array}{c}
3 \\
\end{array} \xleftarrow{1} \begin{array}{c}
2 \\
\end{array}$

W.l.o.g., we can assume $1 \leq k \leq \frac{N}{2}$.

Rem: $P_N^{(k)}$ is connected $\iff \gcd(N, k) = 1$. If $\gcd = d$, it consists of $\frac{N}{d}$ connected components, each isomorphic to $P_{N/d}^{(d)}$. 

\[ \Rightarrow P_{N/d}^{(d)} \]
By construction, vertex 1 (the smallest entry) of $P_N^{(k)}$ is a sink, and mutation $M_1$ turns 1 into a source but leaves the rest of the quiver unchanged. Hence if relabel vertex 1 as $(N+1)$, we see 

$$M_1(P_N^{(k)}) = P_{(2,3,4,...,N+1)}^{(k)} \cong P_N^{(k)} \text{ via cyclic rotation } P$$

i.e., $M_1(P_N^{(k)}) = P(P_N^{(k)})$ as desired.

Claim: Any 1-periodic quiver with the further property that vertex 1 is a sink can be decomposed as a combination, possibly with multiplicities, of $P_N^{(k)}$s.

We can express such a quiver as

$$a_1 P_N^{(1)} + ... + a_{(N+1)} P_N^{(N+1)}$$

where $a_1, ..., a_{(N+1)} \in \mathbb{Z} \geq 0$.

**Example:**

$$3P_6^{(1)} + P_6^{(3)} =$$

![Diagram]

**Proof:** Clearly such a quiver is 1-periodic for the same reason as above. Mutation at 1 turns sink 1 into source $N+1$.

Further, if we begin w/ a quiver that has $\sigma_2$ arrows pointing from vertex $(i+1)$ to vertex 1 [we are assuming 1 is a sink], then 1-periodicity dictates the adjacencies w/ vertex 2.
and inductively, we derive the adjacencies of the remaining vertices
$\Rightarrow$ such a quiver $= \mathbf{c}_1 P_N^{(1)} + \ldots + \mathbf{c}_k P_N^{(k)}$ where $k \leq N/2$.

The Sunus-4 Quiver

\begin{align*}
\begin{array}{c}
 & 1 \\
\downarrow & \downarrow \\
 4 & 3 \\
\end{array}
\end{align*}

is an example of a 1-periodic quiver

where 1 is not a sink.

Thm 6.6 of Fordy-Mars. Any 1-periodic quiver on $N$ vertices can be written as

$$a_1 P_N^{(1)} + a_2 P_N^{(2)} + \ldots + a_N P_N^{(N/2)} + \sum_{l=1}^{N/2-1} \sum_{t=1}^{N/2} E_{ij} \ell+tt P_{(N-2l)}^{(t)}$$

where we let $a_i \in \mathbb{Z}$ (possibly negative)

and $E_{ij}$ is a scalar defined as a primitive with 1 a source etc.

$$E_{ij} := \frac{1}{2} \left( \frac{a_i}{a_j} - \frac{a_j}{a_i} \right)$$

$$= \begin{cases} 
\text{sgn}(a_j) a_i a_j & \text{if } a_i \neq a_j \text{ have opposite signs} \\
0 & \text{if } \text{sgn}(a_i) = \text{sgn}(a_j)
\end{cases}$$

Special case: if all $a_i \geq 0$ [like in case when vertex 1

then $E_{ij} = 0$ for $i < j$.]

Rem: $P_{(N-2l)}^{(t)}$ is a primitive on vertices $(l+1, l+2, \ldots, N-2l)$
E.g., Sumos-4 Quiver $= 2P_4^{(2)} - P_4^{(1)} - 2P_4^{(1)} + Z$

$\varepsilon_{12} = \text{sgn}(+2) \cdot (-1)(2) = -2$

\[ 2P_4^{(2)} - P_4^{(1)} - 2P_4^{(1)} + ZP_2^{(1)} \]

Necessary condition: If $B$ is the exchange matrix associated to a 1-periodic quiver (on $N$ vertices), then $B$'s first row is palindromic (after initial $b_{11}=0$)

E.g., $[b_{12} \ b_{13} \ b_{14} \ b_{15} \ b_{16} \ b_{17}] = [2 \ -1 \ 0 \ 1 \ 0 \ -1 \ 2]$

PF: By Then \((b)\) $b_{1K} = -a_{K-1}$ where $a_{K-1}$ is the coeff of $P_N^{(K)}$ in the decomposition of the 1-periodic quiver.
By def of $P_N^{(K)}$, $b_{1K} = -a_{K-1} = b_{1N-K+2} \Rightarrow$ palindromicity.

As a consequence: First row of $B$ (when palindromic as above) uniquely determines the decomposition.

above e.g. corresponds to $\varepsilon \ P_8^{(1)} - P_8^{(2)} + P_8^{(4)} + ZP_6^{(1)} - P_2^{(2)} P_4^{(1)}$. 
Example: Somos-5 sequence defined by

\[ x_n x_{n+5} = x_{n+1} x_{n+4} + x_{n+2} x_{n+3} \]

If \( x_1 = \ldots = x_5 = 1 \), we get \( x_6, x_7, \ldots = 2, 3, 5, 11, 37, 83, 274, \ldots \)

defined by

\[ P_5^{(2)} - P_5^{(1)} = P_{5-2}^{(1)} \]

Since \( \xi_{12} = 5gn^{(+1)}(-1)(+1) = 1 \)

Notice 2-cycle [2 \( \leftrightarrow \) 4]

\( x_1 x'_1 = x_2 x_5 + x_3 x_4 \) as desired deleted but double arrows 2 \( \rightarrow \) 3 \( \rightarrow \) 4 appear.
Proof of Thm 6.6 (1-Periodic Quivers)

Suppose $Q$ is 1-periodic and has $N \times N$ exchange $B_0$ matrix

We wish to show $\exists \left[ m_{1}, \ldots, m_{N-1} \right] \in \mathbb{Z}^{N-1}$ s.t. $m_k = m_{N-k}$

and $b_{ij} = m_{i-j} + \varepsilon_j \varepsilon_{i-j+1} + \varepsilon_{i-j+2} + \ldots + \varepsilon_{i-j+\ell-1}$ for $i > j$

(recall $\varepsilon_{i-j} = \frac{1}{2} m_{i-j} m_{j-i} - m_{j-i} m_{i-j}$)

Let $J = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \alpha & \beta \\ \vdots & \gamma & \delta \end{bmatrix}$, i.e. the matrix corresponding to cyclic rotation.

$\Rightarrow J(Q)$ associated to $JPB\bar{J}$ (as an exch. matrix).

Let $m_{i-j} = \text{# arrows between } i \text{ and } (i+1)$, sign determines direction $m_{i-j} \Rightarrow -m_{j-i}$

Now suppose that we locally have $i \rightarrow j \rightarrow i$ in $Q$.

Then we have arrows $i \rightarrow j$ in $M_i Q_0$.

Hence for $i > j$ & $i_j \neq 1$, we have $\left[ M_{1} B \right]_{ij} = b_{ij} + \varepsilon_{i-j-1}$

To see this, note that $i \rightarrow j$ is a difference of $(i-1)$

$\Rightarrow i \rightarrow j \Rightarrow (j-1)$

and we have $m_{i-j}$ arrows of the first type $-m_{j-i}$ of the second.

These signs differ to make a 2-path $\Rightarrow \varepsilon_{i-j-1}$ such 2-paths

(Where $\varepsilon_{i-j-1}$ is negative for $i \leftarrow j \rightarrow i$ orientation)
Also, \( \left( P B P^{-1} \right)_{ij} = b_{c-1,j-1} \). (\( \ast \))

Thus \( \mathcal{Q} \) 1-periodic \( \iff \overline{b_{ij} + \varepsilon_{c-1,j-1} = b_{c-1,j-1}} \)
for all \( c > j \) w/ \( c \neq i, j \neq 1 \).

By symmetry, \( b_{c-j+1,j} = m_{c-j,j} \), thus iterating (\( \ast \)) yields (\( \ast \ast \)).

In particular, \( \overline{\varepsilon_{c-1,j-1} = -\varepsilon_{c-1,j-1}} \) and

\[
b \varepsilon_{ij} = b_{c-j+1,j} + \varepsilon_{i-1,j-1} = b_{c-j+1,j} + \varepsilon_{i-2,j-1} + \varepsilon_{i-1,j-1} + \ldots \varepsilon_{i-1,j-1}
\]

\( \implies \)

\[
b \varepsilon_{ij} = m_{c-j+1,j} + 1 \]

Lastly, if we mutate at vertex \( N \) instead, we can run the whole recurrence backwards to get \( m_{k} \leftrightarrow m_{n-k} \) symmetry.