A symplectic manifold is a manifold \( M \) plus "geometric structure" \( \omega \)

(Analogous to a Riemannian manifold with extra geometric structure)

is a metric

Warm-up e.g. A symplectic vector space is a real vector space \( V \) (say of \( \dim d \)) and a non-degenerate 2-form \( \omega \) s.t.

\[
\omega(\vec{v}, \vec{w}) = \vec{v}^T A \vec{w} \quad \text{where } A \text{ is an matrix satisfying } A^T = -A \text{ and } \det A \neq 0.
\]

\[
e.g. \quad J = \begin{pmatrix} O & -I \\ I & O \end{pmatrix} \quad \text{is an example of such a possible } A.
\]

Fact: Every symplectic vec. space is even dim \((d = 2n)\) and has a basis \( \{ p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n \} \) s.t.

\[
\omega(\vec{v}, \vec{w}) = \vec{v}^T J \vec{w}.
\]

Thus \( \omega(\vec{p}_i, \vec{q}_j) = -\omega(\vec{q}_i, \vec{p}_j) = \delta_{ij} \) and \( \omega = dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n \).
Warning: If \( W \) is a subspace of symplectic vec. space \( V \), it is possible for \( W^\perp := \{ \tilde{v} \in V : w(\tilde{v}, \tilde{w}) = 0 \text{ for all } \tilde{w} \in W \} \) to satisfy \( W \cap W^\perp \neq \{0\} \).

In fact, for any \( \tilde{v} \in V \) \( w(\tilde{v}, \tilde{v}) = 0 \) \( \implies \) If we let \( W = \langle \tilde{v} \rangle \), one-dim subspace, \( \langle \tilde{v} \rangle^\perp \supseteq \tilde{v} \) & \( V = \langle \tilde{v} \rangle \oplus \langle \tilde{v} \rangle^\perp \) for e.g.

Subspace \( W \) of symplectic vec space \( V \) is

- **symplectic** if \( W \cap W^\perp = \{0\} \) (i.e. form restricted to \( W \) and \( W^\perp \) each non-degenerate)

- **Lagrangian** if \( W = W^\perp \) (i.e. form \( w \) restricted to \( W \) is zero)

- **isotropic** if \( W = W^\perp \) or

- **co-isotropic** if \( W^\perp = W \).

E.g. For any \( \tilde{v} \in V \) \( \langle \tilde{v} \rangle \) is isotropic.
E.g. any hyperplane (codim 1) is co-isotropic.
e.g., in \( \mathbb{R}^4 \) with coordinates \( p_1, p_2, q_1, q_2 \) and
\[
\mathbf{w} = dp_1 \wedge dq_1 + dp_2 \wedge dq_2,
\]
i.e.,
\[
\mathbf{w}(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{w},
\]
then the 2-planes \( \langle p_1, p_2 \rangle \) and \( \langle q_1, q_2 \rangle \) are Lagrangian, while the 2-planes \( \langle p_1, q_1 \rangle \) and \( \langle p_2, q_2 \rangle \) are symplectic.

\( p_1 \leftrightarrow [1000] \) so
\[
\mathbf{w}(p_1, \mathbf{w}) = [1000] \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_{p_1} \\ w_{p_2} \\ w_{q_1} \\ w_{q_2} \end{bmatrix} = -w_{q_1}
\]

So \( \langle p_1, p_2 \rangle ^\perp = \langle p_1, p_2 \rangle \) since \( \mathbf{w}(p_1, q_1) = -1 \) and \( \mathbf{w}(p_2, q_2) = -1 \).

Lagrangian \( \checkmark \)

Similarly, \( \langle p_1, q_1 \rangle ^\perp = \langle p_2, q_2 \rangle \) since
\[
\mathbf{w}(p_1, \mathbf{w}) = -w_{q_1} \quad \mathbf{w}(q_1, \mathbf{w}) = +w_{p_1}
\]

So symplectic \( \checkmark \).

Rem: Can get all symplectic 2-planes of \( \mathbb{R}^4 \) using Plücker words.

Def: A symplectic manifold "locally looks like" a symplectic (real) vector space.
Def: Given a symplectic manifold \((M, \omega)\) and two smooth functions \(F, G : M \to \mathbb{R}\), the Poisson bracket \(\{F, G\}\) is defined as another function whose value at point \(x\) is

\[
\{F, G\}(x) := \frac{d}{dt} \left. F(P_t(x)) \right|_{t=0}
\]

where \(P_t\) is the Hamiltonian (local) flow of \(X_G\) Hamiltonian vector field.

In particular,

\[
\{F, G\}(x) = \frac{d}{dt} \left. F(x) \right|_{t=0} (X_G(x))
\]

\[
\{F, G\} = -\omega(X_G, X_F) = -\omega(X_F, X_G)
\]

Cor: \(\{G, F\} = -\{F, G\}\).

In the \((p, q)\)-words for \(\mathbb{R}^{2n}\) case,

\[
\{F, G\} := \begin{bmatrix} G_T \quad -G_T \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} F_p \\ -F_q \end{bmatrix}
\]

\[
= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i}
\]
As motivated, an (abstract) Poisson bracket is a skew-symmetric bilinear map \( \{ \cdot, \cdot \} \) satisfying the Leibniz identity \( \{ f_1 f_2, f_3 \} = f_1 \{ f_2, f_3 \} + \{ f_1, f_3 \} f_2 \) and the Jacobi identity

\[ \{ \{ f_1, f_2 \}, f_3 \} + \{ f_2, \{ f_3, f_1 \} \} + \{ f_3, \{ f_1, f_2 \} \} = 0. \]

Def: A Poisson algebra is a commutative associative algebra \( g \) equipped w/ a Poisson bracket \( \{ \cdot, \cdot \} : g \times g \to g \).

Remark: If we let \( g = C^\infty(M) \) the alg. of smooth functions on a symplectic manifold, the usual product \& chain rules of differentiation verify these two identities.

Rem: Given an assoc. alg. \( A \), letting \( \{ x, y \} \) be defined as \( \{ x, y \} = xy - yx \) (the commutator) yields a Lie algebra w/ Lie bracket \( \{ \cdot, \cdot \} \) and together w/ the multi structure of \( A \), this is a Poisson algebra.

Rem: The tensor algebra of a Lie algebra is a Poisson alg.
Def: For $H \in \mathfrak{g}$, if $\int f_j H = 0$, $f_j \in \mathfrak{g}$ is called an integral of the motion $H$.

(thinking of $X_f \in \mathfrak{h}$ as vec. fields)

i.e., $[X_f, X_H] = X - \int f_j H$

Also said that they commute.

Def: An element $c \in \mathfrak{g}$ s.t., $\{c, f\} = 0 \forall f \in \mathfrak{g}$ is called a Casimir element or conserved quantity.

We will see later on the course that Poisson alg's and Casimir elts/conserved quantities have roles in cluster algebra theory.