11/26/18

Following Theorem 3.7 of [Goncharov-Kenyon]

We look at perfect matchings of Γ, a bipartite graph on a torus.

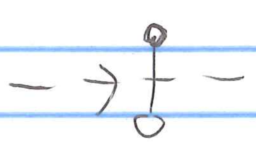
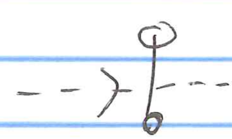
We can obtain Hamiltonians and Casimirs from the data of the perfect Matchings.

First, define the Kastelyn Matrix K_Γ as the weighted adjacency matrix between Black & White vertices.

Entries given a Kastelyn weighting of ± 1 on each edge so that: for every $(4k+2)$-gon face, even # $+1$'s & even # -1's

for every $4k$-gon face, odd # $+1$'s & odd # -1's.

Then give a $z_1^a z_2^b$ weighting of an edge by

 z_i^{+1} and  z_i^{-1} for

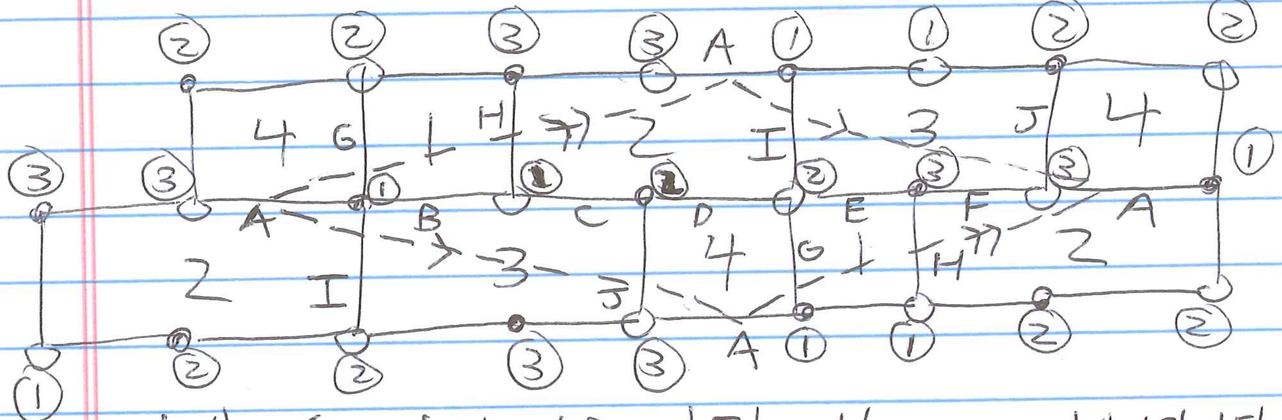
each fundamental cycle α_i of the torus (or surface more generally).

In $\det K_\Gamma$, collect together terms w/ the same $z_1^a z_2^b$ monomial.

Claim: Each of these collections $\longleftrightarrow$ face-twist equivalence class of matching and all have the same signs. (Pf on Wednesday.)

(2)

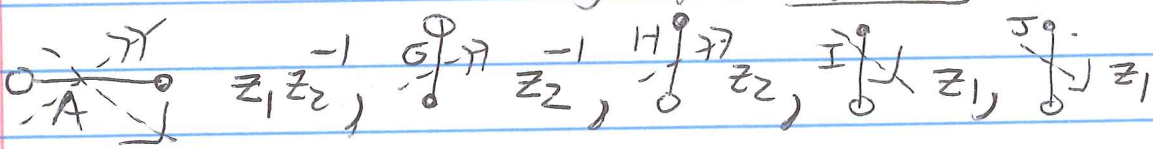
Somos 4 example



$|V| = 6, \ |E| = 10, \ |F| = 4 \Rightarrow \chi = |V| - |E| + |F| = 0$

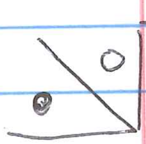
(a torus of $g=1$)

Kasteleyn weighting by letting edge G get sgn (-1) and the remaining edges sgn $(+1)$.



A: $z_1 z_2^{-1}$, G: z_2^{-1}, H: z_2, I: z_1, J: z_1

Kasteleyn matrix has determinant

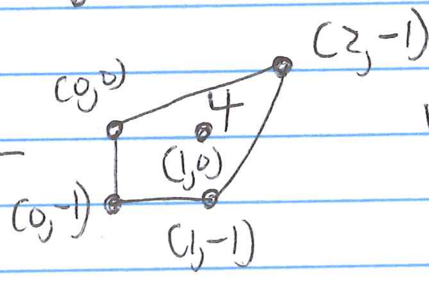


$$\begin{bmatrix} B & I z_1 - G z_2^{-1} & A z_1 z_2^{-1} \\ C & D & J z_1 \\ H z_2 & E & F \end{bmatrix}$$

$$\begin{aligned} & HIJ\, z_1^2 z_2 \\ & + BDF \\ & + ACE\, z_1 z_2^{-1} \\ & + CFG\, z_2^{-1} \\ & - (ADH + CFI + BEJ + GHJ)\, z_1 \end{aligned}$$

Each of these terms is a Matching of Γ.

Toric Diagram or Newton Polygon



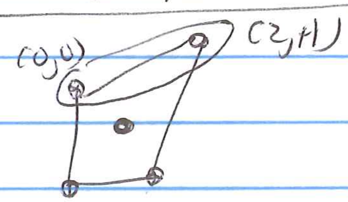
using exponent vectors of $\det K_\Gamma$.

(3) For internal points of the toric diagram, i.e. collecting terms w/ same $z_1^a z_2^b$ factor, corresponding polynomial

is a Hamiltonian and choosing one matching as a base, all other terms/matchings reachable from the base by a sequence of face twists.
we record these with $X_1, \ldots, X_{|F|-1}$.
we don't need to use $X_{|F|}$ since

$$X_{|F|} = X_1^{-1} X_2^{-1} \cdots X_{|F|-1}^{-1}.$$

Ratios of singleton orbits (under face twist equivalence) appearing as extremal & external vertices making up a side of the toric diagram

e.g. 

Correspond to Casimirs or zig-zag paths in Γ.

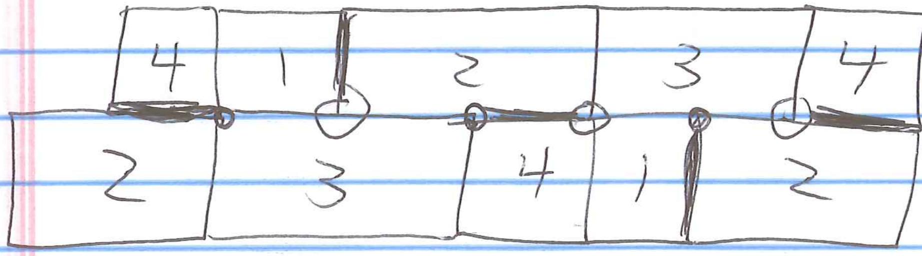
E.g. in Sumos 4 e.g., there is a single Hamiltonian

corresponding to $z_1(ADH + CFI + BEJ + GHJ)$

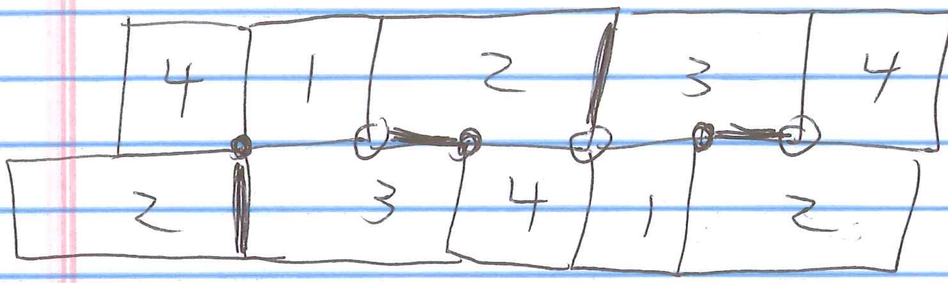
$$\leftrightarrow z_1(X_1X_2X_3 + X_1X_3 + X_1 + 1)$$

Matchings corresponding to Internal lattice points of same $z_1^a z_2^b$ sum up to a Hamiltonian

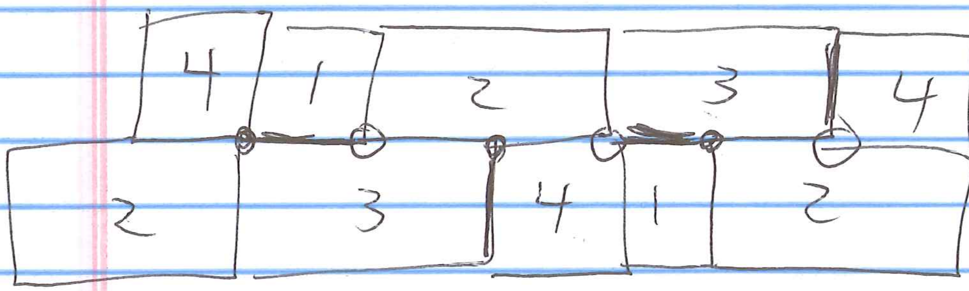
4



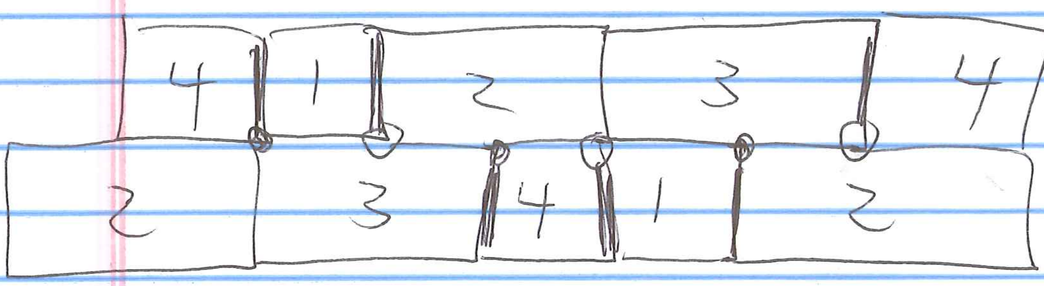
ADH



CFI



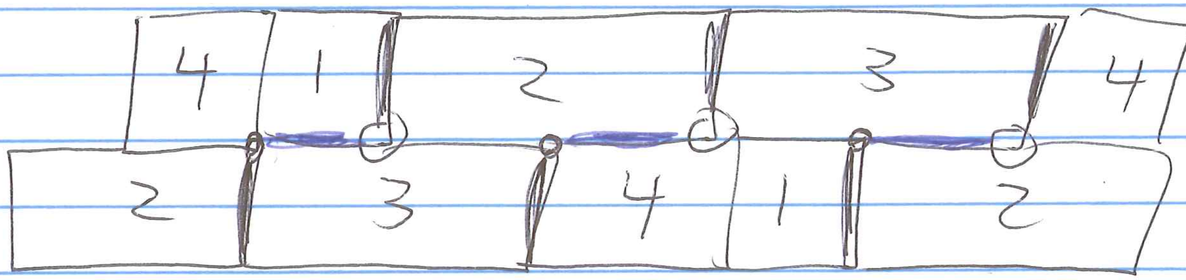
BEJ



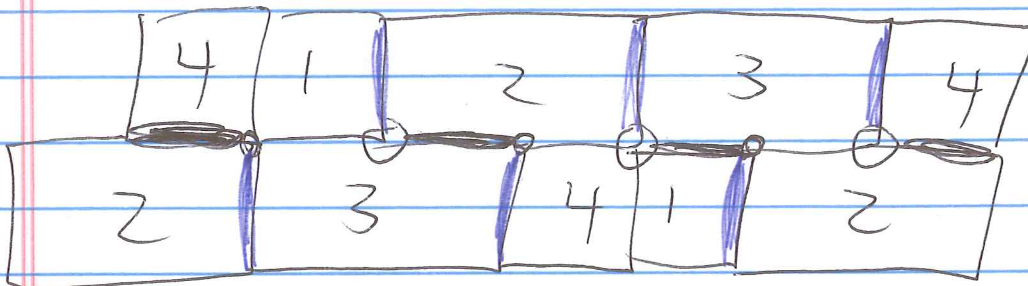
GHJ

5

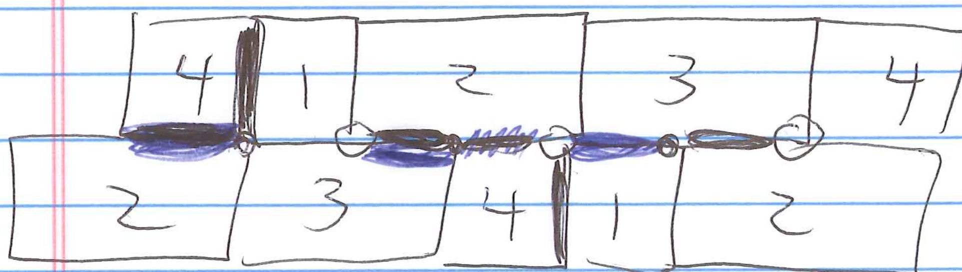
Ratios of Matchings corresponding to extremal external lattice points are Casimirs & zig-zag paths



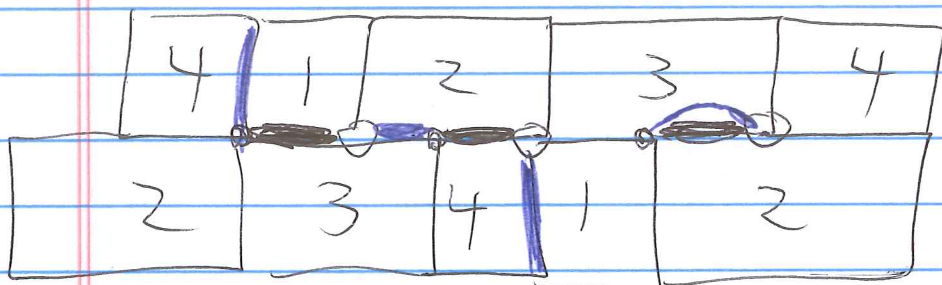
$$\frac{HIJ\, z_1^2 z_2}{BDF\, z_1^0 z_2^0}$$



$$\frac{ACE\, z_1 z_2^{-1}}{HIJ\, z_1^2 z_2}$$



$$\frac{CFG\, z_1^0 z_2^{-1}}{ACE\, z_1 z_2^{-1}}$$



$$\frac{BDF\, z_1^0 z_2^0}{CFG\, z_1^0 z_2^{-1}}$$