Following Theorem 3.7 of [Goncharov-Kenyon]

We look at perfect matchings of $\Gamma$, a bipartite graph on a torus.

We can obtain Hamiltonians and Casimirs from the data of the perfect matchings.

First, define the Kasteleyn Matrix $K_{\Gamma}$ as the weighted adjacency matrix between black and white vertices.

Entries given a Kasteleyn weighting of $\pm 1$ on each edge so that: for every $(4k+2)$-gon face, even # $+1$'s and even # $-1$'s,

For every $4k$-gon face, odd # $+1$'s and odd # $-1$'s.

Then give a $z^a z^b$ weighting of an edge by

$\rightarrow \frac{\partial}{\partial z^a} - z^a \rightarrow \frac{\partial}{\partial z^b} - z^b$ for each fundamental cycle $\alpha_i$ of the torus (or surface more generally).

In $\det K_{\Gamma}$, collect together terms with the same $z^a z^b$ monomial.

Claim: Each of these collections $\leftrightarrow$ face-twist equivalence class of matching ad all have the same signs. (PF on Wednesday.)
Sommus 4 example

\[ |V| = 6, \ |E| = 10, \ |F| = 4 \implies x = |V| - |E| + |F| = 0 \]

(a term of \( g = 1 \))

Kasteleyn weighting by letting edge \( G \) get \( \text{sgn} (-1) \)
and the remaining edges \( \text{sgn} (+1) \).

Kasteleyn matrix has determinant

\[
\begin{vmatrix}
B & I_{z_j} - G_{z_j}^{-1} A_{z_j} z_{z_j}^{-1} \\
0 & I_{z_j} - J_{z_j} z_{z_j}^{-1}
\end{vmatrix}
\]

\[ + BDF + \text{ACE} z_{z_j} z_{z_j}^{-1} + \text{CFG} z_{z_j}^{-1} \]

Each of these terms is a matching of \( M \).

Toric Diagram

Newton Polygon
For internal points of the toric diagram, i.e., collecting terms with same \( z_1^aqz_2^b \) factor, the corresponding polynomial is a Hamiltonian and choosing one matching as a base, all other terms/matchings reachable from the base by a sequence of face twists, we record these with \( X_{\gamma} \rightarrow X_{\gamma \rightarrow 1} \). We don't need to use \( X_{\gamma \rightarrow 1} \) since

\[
X_{\gamma \rightarrow 1} = X_{\gamma}^{-1} X_{\gamma_2}^{-1} \cdots X_{\gamma_{\gamma \rightarrow 1}}^{-1}
\]

Ratios of singleton orbits (under face twist equivalence) appearing as extremal & external vertices making up a side of the toric diagram

E.g., correspond to Casimirs or zig-zag paths in \( \Gamma \).

E.g., in sumos 4 e.g., there is a single Hamiltonian corresponding to

\[
\mathbb{Z}_1 \left( \text{ADH} + \text{CFI} + \text{BEJ} + \text{GHJ} \right)
\]

\[\leftrightarrow \mathbb{Z}_1 \left( X_1X_2X_3 + X_1X_3 + X_1 + 1 \right)\]
Matchings corresponding to internal lattice points of same $\mathbb{Z}_1 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ sum up to $\mathcal{H}$.
Ratios of Casimirs & Zig-Zag paths

Matchings corresponding to extremal external lattice points

\[ \text{HJ} \sim \mathbb{Z}_2^2 \]
\[ \text{BDF} \, \mathbb{Z}_1^2 \]
\[ \text{ACE} \sim \mathbb{Z}_2 \]
\[ \text{HJ} \sim \mathbb{Z}_2^2 \]

\[ \text{CFG} \sim \mathbb{Z}_1^2 \]
\[ \text{ACE} \sim \mathbb{Z}_2 \]

\[ \text{BDF} \, \mathbb{Z}_1^2 \]
\[ \text{CFG} \sim \mathbb{Z}_2 \]