Teichmüller Theory and the Weil–Petersson Form

(See Sec 6.2 of [GSV10], Lectures 26 & 27 from 2011, Lectures 4 & 5 from MSR11.)

Given a marked surface \((S, M)\), we let

\( \mathcal{J}(S, M) \) denote the Teichmüller space defined as

the space of hyperbolic metrics on \((S, M)\)

(\text{constant curvature } -1)

that

- has geodesic boundary on \( \mathcal{E}S \)
- has cusps at marked pts of \( M \)

(Elements of \( \mathcal{J}(S, M) \) considered up to \( \text{Diff}^0(\text{diffeomorphisms isopic to identity}) \))

Fact: \( \mathcal{J}(S, M) \) is a real manifold with dimension

\[ 6g - 6 + 2p + 3b + c \]

\( g = \text{genus } (S) \), \( b = \# \text{ components of } \mathcal{E}S \)

\[ p + c = |M|, \text{ i.e. } c = |M \setminus \mathcal{E}S|, \text{ i.e. punctures} \]

\[ p = |\text{marked pts on boundary}|, \text{ i.e. marked pts} \]

Example: \((n+3)\)-gon w/ \( g=0, b=1, p=0, c=n+3 \)

Claim: \( \mathcal{J}(S, M) \) has dim equal to \( n \) in this case.

We view this polygon being given a choice of hyperbolic metric.
View this polygon as the Poincaré disk model for hyperbolic space with metric given by

\[ ds^2 = \frac{dx^2 + dy^2}{(1-r^2)^2}, \quad r = \sqrt{x^2 + y^2} \]

According to this metric, geodesics (curves from A to B with the shortest length) lie along diameters or are arcs perpendicular to the boundary.

Further, points on the boundary are infinitely far away from the center:

Distance from \((0,0)\) to \((1,0)\) is

\[
\int_0^1 \frac{dx}{1-x^2} = \left[ \tanh^{-1}(x) \right]_0^1 = \frac{1}{2} \log \left( \frac{1+x}{1-x} \right) \bigg|_0^1 \rightarrow \infty.
\]

"Faster to travel closer to center" hence shorter to move inward and back outward than Euclidean line path.

For \((n+3)\)-gon, a point in \(\mathcal{F}(S,M)\) is a choice of a map \((n+3)\)-gon \(\rightarrow\) Poincaré disk s.t.
$(n+3)$-gon $\quad \longrightarrow \quad$ Poincaré disk

Marked pts $\longrightarrow$ pts on "boundary" at $\infty$ of disk

boundary segments $\longrightarrow$ geodesics between those pts on circle at $\infty$

internal digonals $\longrightarrow$ other geodesics

E.g.,

We consider two points in $\mathcal{F}(5,\mathbb{M})$ to be the same if they agree up to diffeomorphisms (including rotations, dilations, etc.)

Thus don't get to choose $a_j = a_{n+3}$ as independent real coordinates.

Instead: thinking of $\infty$-circle $\quad \longrightarrow \quad$ as $\mathcal{R}(1)$

Up to diffeo., can assume

$\quad a_1 = \infty, \quad a_2 = -1, \quad a_3 = 0 \quad [\text{some references use } \infty, 0, 1 \text{ instead}]$
In fact, then we do get to choose $a_4, \ldots, a_{n+3}$ freely (Hence $\mathcal{F}(S,M) \cong \mathbb{R}^n$) in this case.

This is up to the action of $\text{PSL}_2(\mathbb{R}) = \left\{ \frac{a x + b}{c x + d} \right\}/\lambda$, orientation-preserving Mobius transformations, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 1$.

**Def:** Given a triangulation $T = \{E_i\}_{i=1}^n$ of (internal) arcs of $(S,M)$ and a choice of hyperbolic structure $\Sigma \in \mathcal{F}(S,M)$ by considering the diffeomorphism (or action of $\text{PSL}_2(\mathbb{R})$) that maps the quadrilateral inscribing $E_i$ to we define the shear coordinate

$$\sum(E_i \cdot T)$$

to be value $x$.

e.g., $(n+3)$-gon has $n$ internal arcs and yields $n$ shear coordinates.

$$6g - 6 + 2p + 3b + c$$

shear coordinates for a general $(S,M)$.

(Freely chosen)
More precisely (in presence of punctures):

**Theorem:** The map \( \mathcal{T}(S_j M) \to \mathbb{R}^n \)

\[
\sum_{i=1}^{\# E_i \text{ in } T} \gamma_{E_i} (E_{ij} T) = \frac{1}{|k|} \prod_{j=1}^{K} \gamma_{E_{ij}} (E_{ij} T) = 1
\]

This relation explains discrepancy in the cont & dim

If we change the triangulation by Flips:

\[
\mathcal{T} \to \mathcal{T}'
\]

Then shear coordinates change as follows:

\[
\gamma_{E'} (E'_{ij} T') = \gamma_{E} (E_{ij} T)^{-1}
\]

\[
\gamma_{A'} (A'_{ij} T') = \gamma_{A} (A_{ij} T)(1 + \gamma_{E} (E_{ij} T))^{-1}
\]

\[
\gamma_{B'} (B'_{ij} T') = \gamma_{B} (B_{ij} T)(1 + \gamma_{E} (E_{ij} T))^{-1}
\]

\[
\gamma_{C'} (C'_{ij} T') \text{ analogous to } \gamma_{A} (A_{ij} T)
\]

\[
\gamma_{D'} (D'_{ij} T') \text{ analogous to } \gamma_{B} (B_{ij} T)
\]
Given a triangulation $T$ of $(S, \Sigma)$ such that
- there are no self-folded triangles (i.e., edges of any triangle are pairwise distinct)
- and all vertices have at least three incident edges (i.e., punctured bigon disallowed)

called a perfect triangulation in textbooks.

We can define the Weil-Petersson form associated to $T$
\[
W = \sum_{\text{pairs of edges } A, B \text{ in } T} \frac{dF(A) \wedge dF(B)}{F(A) \cdot F(B)}
\]
where $F : T \to \mathbb{R}$ is a coordinate.

We will discuss decorated Teichmüller space, lambda lengths, and cords. Up to a constant, this is the unique closed 2-form on the cluster manifold $\mathcal{Y}(\Sigma)$ that is compatible with the associated cluster algebra $\mathcal{A}(\Sigma)$.

Claim: We can rewrite $W$ as
\[
W = \frac{1}{2} \sum_{E \in T} (dx_{x_1} \wedge dx_{E} - dx_{x_2} \wedge dx_{E} + dx_{x_3} \wedge dx_{E} - dx_{x_4} \wedge dx_{E})
\]
where $x_{E} := \log F(E)$ & $E$ inscribed in the quadrilateral.