10/5/18

Teichmüller Theory and the Weil-Petersson form

(See Sec 6.2 of [GSV10], Lectures 26 & 27 from 2011, Lectures 4 & 5 from MSRI '11.)

Given a marked surface (S, M), we let

$\mathcal{T}(S,M)$ denote the Teichmüller space defined as

the space of hyperbolic metrics on (S,M)
(constant curvature -1)
that

- has geodesic boundary on ∂S
- has cusps at marked pts of M.

(Elements of $\mathcal{T}(S,M)$ considered up to Diffeo^{0} (diffeomorphisms isotopic to identity))

Fact: $\mathcal{T}(S,M)$ is a real manifold with dimension
$6g-6+2p+3b+c$

$g = \mathrm{genus}(S)$, $b = \#$ components of ∂S,
$p+c = |M|$, i.e. $c = |M \cap \partial S|$, $p = |M \cap (S - \partial S)|$
marked pts on boundary; interior marked pts, i.e. punctures

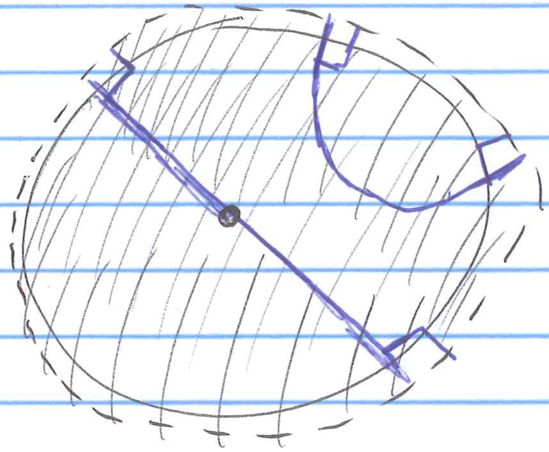
Example: $(n+3)$-gon w/ $g=0$, $b=1$, $p=0$, $c=n+3$

Claim: $\mathcal{T}(S,M)$ has dim equal to n in this case.

We view this polygon being given a choice of hyperbolic metric.

(2) View this polygon as Poincaré disk model for hyperbolic space w/ metric given by

ds such that $ds^2 = \dfrac{dx^2 + dy^2}{(1-r^2)^2}$ w/ $r = \sqrt{x^2+y^2}$



According to this metric, geodesics (curves from A to B w/ the shortest length) lie along diameters or are arcs $\perp$ boundary.

- Further, points on the boundary are ∞-ly far away from the center:

distance from $(0,0)$ to $(1,0)$ is $\displaystyle\int_0^1 \frac{dx}{1-x^2}$

$$= \operatorname{arctanh}(x)\Big|_0^1 = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\Big|_0^1 \to \infty.$$

- "Faster to travel closer to center" hence shorter to move inward and back outward than Euclidean line path.

For $(n+3)$-gon, a point in $\mathcal{T}(S,M)$ is a choice of a map

$(n+3)\text{-gon} \longmapsto \text{Poincaré disk}$ s.t.

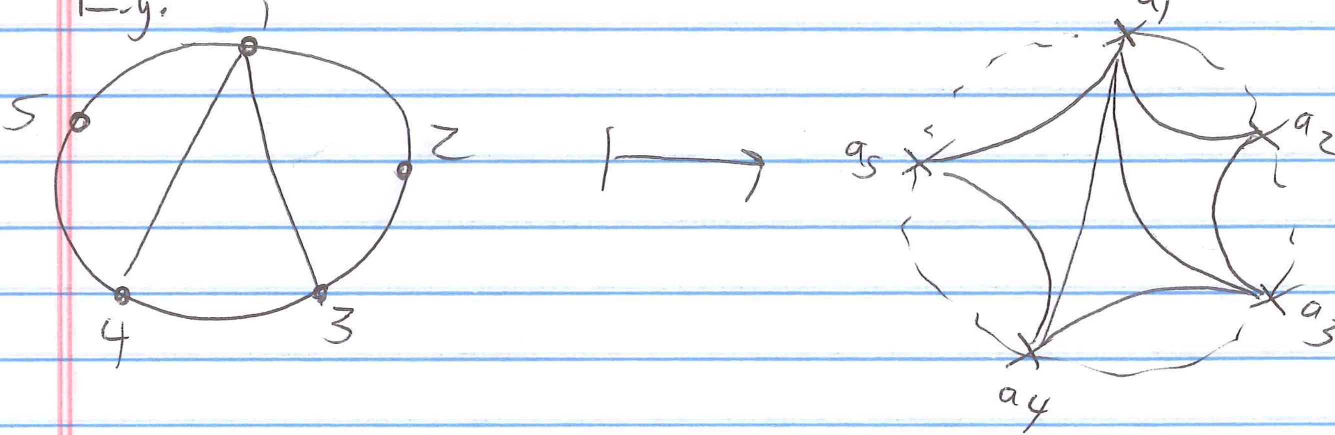
$(n+3)$-gon $\longmapsto$ Poincaré disk

(3) Marked pts $\longrightarrow$ pts on "boundary" at ∞ of disk

boundary segments $\longrightarrow$ geodesics between these chose pts on circle at ∞

internal diagonals $\longrightarrow$ other geodesics

E.g.

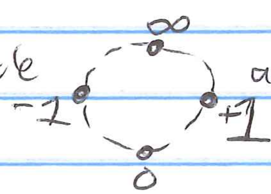


We consider two points in $\mathcal{T}(S,M)$ to be the same if they agree up to Diffeomorphisms isotopic to the identity

(including rotations, dilations, etc.)

Thus don't get to choose $a_1, \dots, a_{n+3}$ as independent real coordinates.

Instead: thinking of ∞-circle



as $\mathbb{R} \cup \{\infty\}$

up to diffeo, can assume

$a_1 = \infty$, $a_2 = -1$, $a_3 = 0$ [some references use $\infty, 0, +1$ instead]

(4)

In fact, then we do get to choose $a_4, \ldots, a_{n+3}$ Freely $\left(\text{Hence } \mathcal{T}(S,M) \cong \mathbb{R}^n \text{ in this case}\right)$

This is up to the action of $PSL_2(\mathbb{R}) = \left\{ \frac{ax+b}{cx+d} \right\} / \lambda$

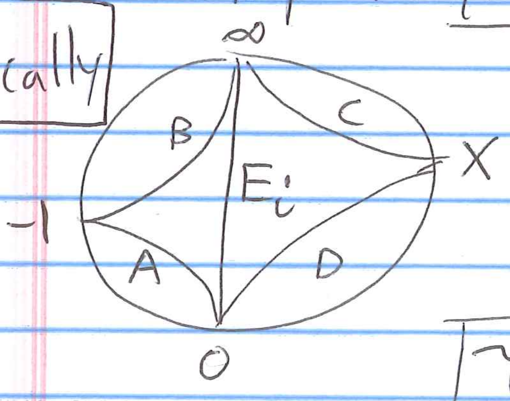
orientation-preserving Mobius transformations

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$

<u>Def:</u> Given a <u>triangulation</u> $T = \{E_i\}_{i=1}^{n}$ of (internal) arcs of (S,M) and a choice of <u>hyperbolic structure</u> $\Sigma \in \mathcal{T}(S,M)$,

by considering the diffeomorphism (or action of $PSL_2(\mathbb{R})$) that maps the <u>quadrilateral inscribing E_i</u> to

$\boxed{\text{locally}}$



we define the <u>shear coordinate</u>

$\boxed{\tau_\Sigma(E_i ; T) \text{ to be value } X}$

e.g. $(n+3)$-gon has n internal arcs and yields n shear coordinates.

$(6g-6+2p+3b+c)$ shear coordinates for a general (S,M).
(Freely chosen)

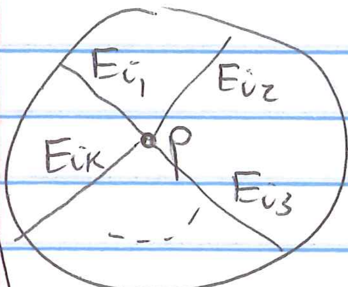
More precisely (in presence of punctures):

Theorem: The map $\mathcal{T}(S,M) \longrightarrow \mathbb{R}^n$

$$\Sigma \longmapsto \{\tau_\Sigma(E_i;T)\}_{i=1}^n$$

$\# E_i$ in T $= 6g-6+3\underline{p}+3b+c$

is a homeomorphism onto the subset of $\mathbb{R}^n$ where for each puncture p with incident arcs $E_{i_1}, \dots, E_{i_K}$, we have

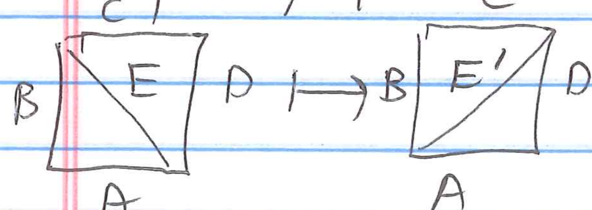


$$\boxed{\prod_{j=1}^{K} \tau_\Sigma(E_{i_j};T) = 1}$$

This relation explains discrepency in the count & dim

If we change the triangulation by flips:

$T \longrightarrow T'$, then shear coordinates change as follows:



(Flip called a <u>Whitehead move</u>)

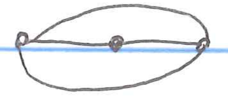
Compare w/ τ-coordinate transformations

$$\left\{\begin{aligned} \tau_\Sigma(E';T') &= \tau_\Sigma(E;T)^{-1} \\ \tau_\Sigma(A;T') &= \tau_\Sigma(A;T)\left(1+\tau_\Sigma(E;T)^{-1}\right)^{-1} \\ \tau_\Sigma(B;T') &= \tau_\Sigma(B;T)\left(1+\tau_\Sigma(E;T)\right) \end{aligned}\right.$$

and $\tau_\Sigma(C;T')$ analogous to $\tau_\Sigma(A;T')$ | $\tau_\Sigma(D;T')$ analogous to $\tau_\Sigma(B;T')$

(6)

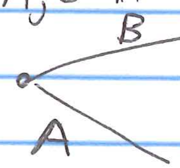
Given a triangulation T of (S, M)
such that
- there are no self-folded triangles 
  (i.e. edges of any triangle are pairwise distinct)

and
- all vertices have at least three incident edges
  (i.e. punctured bigon  disallowed)

[called a <u>perfect</u> triangulation in textbook]

we can define the weil-peterssun form associated to T

as $$w = \sum_{\substack{\text{pairs of} \\ \text{edges } A, B \text{ in } T}} \frac{dF(A)}{F(A)} \wedge \frac{dF(B)}{F(B)}$$

s.t. 

where $F: T \to \mathbb{R}$
$E \in T \mapsto F(E)$
coordinate

We will discuss decorated Teichmüller space, lamba lengths, and coords F more next time.

Up to a constant, this is the unique closed 2-form on the cluster manifold $\mathcal{X}(\Sigma)$ that is compatible with the associated cluster algebra $\mathcal{A}(\Sigma)$.

<u>Claim:</u> We can rewrite w as

$$w = \frac{1}{2} \sum_{\substack{E \in T \\ \text{(internal arcs)}}} \left(dx_{e_1} \wedge dx_E - dx_{e_2} \wedge dx_E + dx_{e_3} \wedge dx_E - dx_{e_4} \wedge dx_E \right)$$

where <u>$x_e := \log F(e)$</u> & E inscribed in the quadrilateral