10/19/18 Last time we discussed that shear coordinates transform under Quadrilateral Flips (of edge E_i) as

$$b_L(E_j';T') = \begin{cases} -\,b_L(E_i;T) & \text{if } j=i \\ b_L(E_j;T) + b_{ij}\cdot\left(b_L(E_i;T) \oplus 0\right) & \text{if } b_{ij}>0 \\ b_L(E_j;T) - b_{ij}\cdot\left(-b_L(E_i;T) \oplus 0\right) & \text{if } b_{ij}<0 \\ b_L(E_j;T) & \text{if } b_{ij}=0 \end{cases}$$

where $(a \oplus b) = \max(a,b)$

$\mathrm{Trop}(1) = 0$

$\mathrm{Trop}(\times) = +$

$\mathrm{Trop}(+) = \oplus$

Tropical analogue of τ-coord/cross-ratio transforms

Today we similarly define a tropical version of λ-lengths.

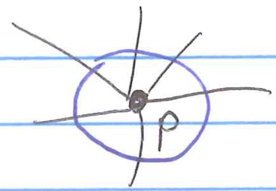
For any arc or boundary segment γ and certain Laminations L (Bounded, i.e. no spiralling into punctures)

we define $a_L(\gamma) := \dfrac{\#\text{ crossings w/ } L \text{ w/ } \gamma}{2}$.

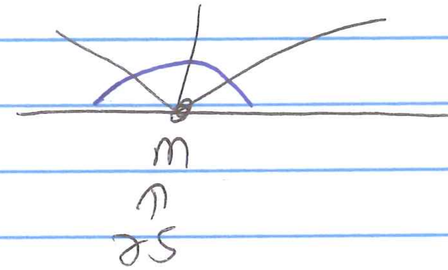
Rem: For technical reasons, we focus only on unpunctured marked surfaces (S,M) today.

Would need to use "openned surfaces" to deal w/ otherwise # of infinite crossings between L & γ if involving punctures.

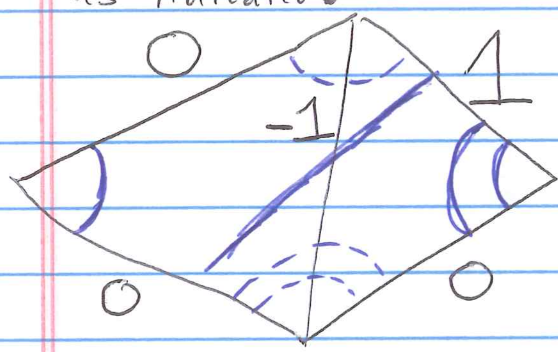
(2) Furthermore, we allow our laminations to have <u>negatively-valued</u> weights on elem. laminations that were previously disallowed, e.g.,

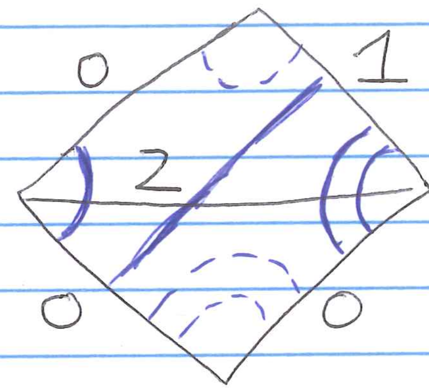


or



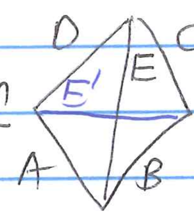
<u>E.g.</u> Given Lamination L on a Quadrilateral, with some negative weights (denoted as <u>dashed</u> curves), we obtain $a_L(\gamma)$'s as indicated.





Note that we can flip $\underset{\text{vertical}}{E} \longrightarrow \underset{\text{horizontal}}{E'}$ w/ same lamination and get <u>a new $a_L(E')$</u> but other values unchanged.

<u>Claim:</u> $a_L(\gamma)$'s satisfy <u>tropical Ptolemy Relation</u>



$$a_L(E) + a_L(E') = \max\left(a_L(A) + a_L(C),\ a_L(B) + a_L(D)\right).$$

<u>Claim: Relation between $b_L(\gamma)$'s & $a_L(\gamma)$'s as tropical cross-ratio</u>

$$b_L(E) = a_L(A) - a_L(B) + a_L(C) - a_L(D).$$

(3)

E.g. continued

$-1+2 = \max(0+1, 0+0)$ ✓

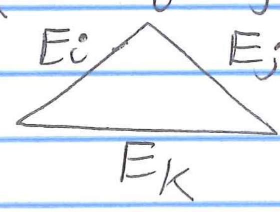
$b_L(E;T) = +1 = 0-0+1-0$ ✓

$b_L(E';T) = -1 = 0-0+0-1$ ✓

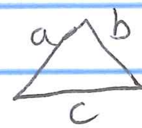
Additionally, can reconstruct L from arbitrary $\vec{a} \in \mathbb{Z}^{n+c}$
$(n = |T|,\ c = \#$ boundary segments$)$
so that $a_L(E_i) = a_i$ for each $i \in \{1, 2, \ldots, n+c\}$.

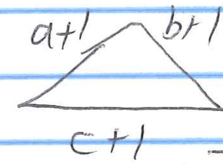
Step 1 (Assuming no self-folded triangles) For any $\vec{a} \in \mathbb{Z}^{n+c}$, there exists $N \gg 0$ s.t. the triangle inequality

$$|\tilde{a}_i - \tilde{a}_j| \leq \tilde{a}_K \leq \tilde{a}_i + \tilde{a}_j \quad [\text{where } \tilde{a}_i = a_i + N\ \forall i]$$

satisfied for each 

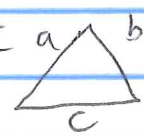
in $T \cup \{$Boundary Arcs$\}$.

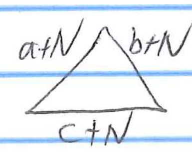
Firstly, once 

satisfies the triangle inequality, so does 

.

$$\left[|a-b| \leq c \leq a+b \Rightarrow |(a+1)-(b+1)| = |a-b| \leq c+1 \leq (a+1)+(b+1) = (a+b+1)+1 \right]$$

And if 

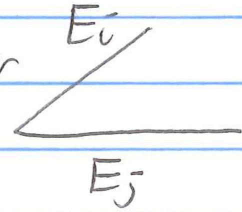
doesn't satisfy triangle inequality, 

will for large enough N, since spread in inequalities widening.

④

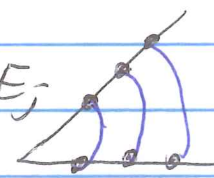
Step 2:

Given $\tilde{a}_i$'s $[\tilde{a}_i = a_i + N \ \forall i]$ such that triangle inequalities satisfied, we place $2\tilde{a}_i$ points along each E_i.

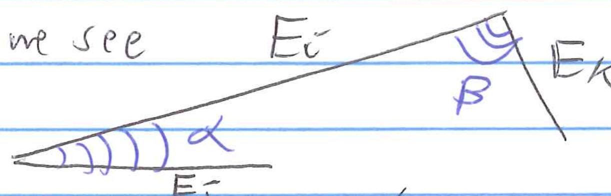
For each corner 

opposite edge 

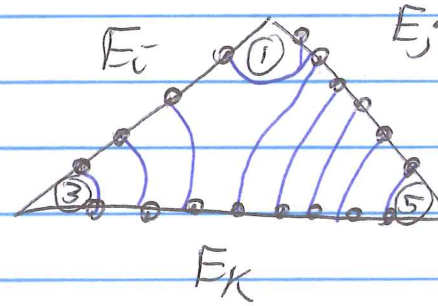
,

connect $(\tilde{a}_i + \tilde{a}_j - \tilde{a}_K)$ marked points on E_i & E_j 

Triangle inequality $\Rightarrow$ this value ≥ 0 and leads to the unique configuration.

Then along edge E_i, we see 

$$2\tilde{a}_i = \alpha + \beta$$
$$= (\tilde{a}_i + \tilde{a}_j - \tilde{a}_K) + (\tilde{a}_i + \tilde{a}_K - \tilde{a}_j) \quad \boxed{\checkmark}$$

E.g. $\tilde{a}_i = 2$, $\tilde{a}_j = 3$, $\tilde{a}_K = 4$ 

$\tilde{a}_i + \tilde{a}_j - \tilde{a}_K = 1$

$\tilde{a}_i + \tilde{a}_K - \tilde{a}_j = 3$

$\tilde{a}_j + \tilde{a}_K - \tilde{a}_i = 5$

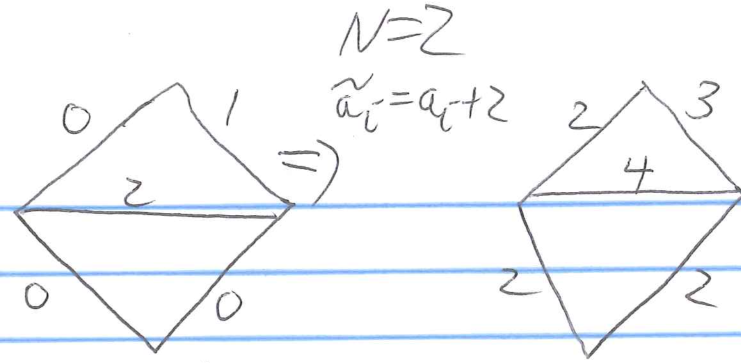
We glue together full triangulation in this way.

Step 3:

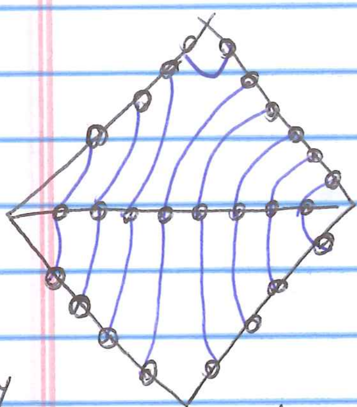
Lastly, we add in "negative" elem. laminations associated to boundary segments 
to cancel out the parameter of N added to achieve triangle inequality everywhere.

Positive & negative elem. laminations cancel each other out.

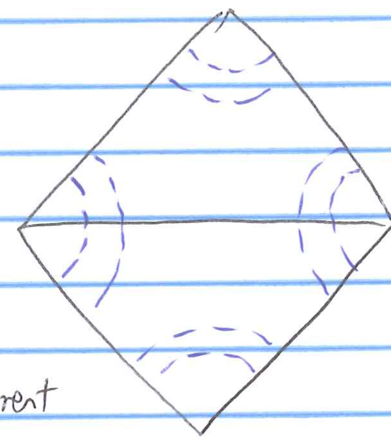
(5) **Example:** $\vec{a} =$



Building lamination according to step 2

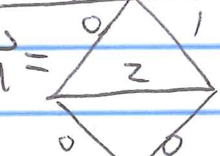


Add in $\longrightarrow$ $-N=-2$ for each boundary segment

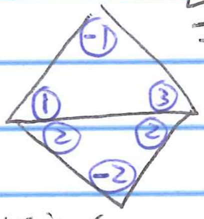


Equivalently

$\vec{a} =$



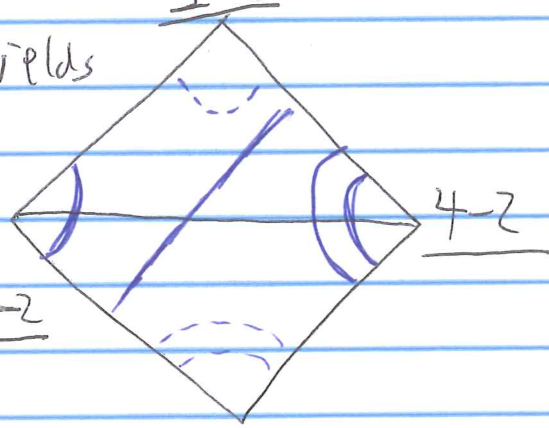
yields



using $(a_i + a_j - a_k)$'s

yields

Locally $=$



agreeing w/ our running example.

We now switch topics and next week see how cluster algebras relate to discrete integrable systems.

We begin with Y-system Dynamics. Historically, this was one of the first applications of Cluster Algebras and also is a natural bridge from our discussion of τ- and cross-ratio coordinates.

See Sections 3, 3 or 6 of [GR18]

or Sections 3.3, 3.5, 3.6 of [FWZ].

6

We can define Y-system associated to every cluster mimicing τ- or cross ratio coordinates

Def: For $n \times n$ B (or Q) initialize $\{Y_1, Y_2, \dots, Y_n\}$. Then define Y-mutation by (in kth direction)

$$Y_j' = \begin{cases} Y_k^{-1} & \text{if } j=k \\ Y_j \cdot (1+Y_k)^{b_{jk}} & \text{if } j\neq k \;\&\; b_{jk}>0 \\ Y_j \cdot \left(1+\frac{1}{Y_k}\right)^{b_{jk}} & \text{if } j\neq k \;\&\; b_{jk}<0 \\ Y_j \cdot & \text{if } j\neq k \;\&\; b_{jk}=0 \end{cases}$$

Rem: Equivalent to write $\boxed{Y_j' = Y_j\, Y_k^{[-b_{jk}]_+}(1+Y_k)^{b_{jk}} \text{ for } j\neq k}$

since $\frac{1+Y_k}{Y_k} = \left(1+\frac{1}{Y_k}\right)$ (for $b_{jk}<0$)

Claim: Given $(n+m)$-by-n extended exchange matrix $\tilde{B}$ and letting $\hat{Y}_j = \prod_{i=n+1}^{n+m} x_i^{b_{ij}}$, then

$$\mu_k\left(\left(\hat{Y}_1, \dots, \hat{Y}_n\right), B\right) = \left(\left(\hat{Y}_1', \dots, \hat{Y}_n'\right), B'\right)$$

given by Y-mutation

where we define $\hat{Y}_j'$ using new cluster var x_k' & exchange matrix B'.

PF: Essentially the τ-coordinate change-of-basis from earlier.

7. Zamolodchikov Periodicity (Conjecture until Fomin-Zelevinsky '02)

Let Q be an oriented Dynkin Diagram (oriented in alternating or bipartite fashion)

e.g., A_n $\circ \to \circ \leftarrow \circ \to \circ \leftarrow \circ$ $\quad E_6, E_7, E_8$

B_n/C_n $\circ \xrightarrow{(2,1)} \circ \leftarrow \circ \to \circ \leftarrow \circ$ $\quad F_4$ $\circ \to \circ \xleftarrow{(2,1)} \circ \to$

D_n $\begin{matrix} \circ \searrow & & & & & & \\ & \circ & \leftarrow \circ & \to & \circ & \leftarrow & \circ \\ \circ \nearrow & & & & & & \end{matrix}$ $\quad G_2$ $\circ \xrightarrow{(3,1)} \circ$

Color vertices black/white if source/sink.

Mutate at all sources first, then all sinks, then iterate

yields $$Y_{i,t+1}\,Y_{i,t-1} = \prod_{j \neq i} \left(Y_{j,t} + 1\right)^{-a_{ij}}$$ is periodic.

(where $[a_{ij}]$ = Cartan matrix of assoc. root system)

i.e. $a_{ij} = -|b_{ij}|$ for $i \neq j$

Periodic of order $2(h+2)$ w/ h = Coxeter # assoc. to Dynkin Diagram/root system.

E.g. $Q = 1 \to 2$ (Type A_2) $\mu_1 \circ \mu_2 \circ \mu_1 \circ \mu_2 \circ \cdots$

($h=3$ in this case)

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$Y_{1,t}$	Y_1	Y_1^{-1}	$\frac{Y_2}{Y_1Y_2+Y_1+1}$	$\frac{Y_1Y_2+Y_1+1}{Y_2}$	Y_2^{-1}	Y_2
$Y_{2,t}$	Y_2	$\frac{Y_1Y_2}{Y_1+1}$	$\frac{Y_1+1}{Y_1Y_2}$	$\frac{1}{Y_1(Y_2+1)}$	$Y_1(Y_2+1)$	Y_1

(8) We see $Y_{1,\text{odd}}\, Y_{1,\text{odd}+2} = \left(Y_{2,\text{odd}+1}^{+1}\right)^{+1}$

$$Y_{2,\text{even}}\, Y_{2,\text{even}+2} = \left(Y_{1,\text{even}+1}^{+1}\right)^{+1}$$

$$Y_{1,\text{even}}\, Y_{1,\text{even}+2} = \left(Y_{2,\text{even}+1}^{-1} + 1\right)^{-1}$$

$$Y_{2,\text{odd}}\, Y_{2,\text{odd}+2} = \left(Y_{1,\text{odd}+1}^{-1} + 1\right)^{-1}$$

e.g.

$$Y_{1,1}\, Y_{1,3} = \left(\frac{1}{Y_1}\right)\left(\frac{Y_1 Y_2 + Y_1 + 1}{Y_2}\right) = \left(Y_{2,2} + 1\right)$$

$$Y_{2,2} = \frac{Y_1 + 1}{Y_1 Y_2}$$

$$Y_{2,0}\, Y_{2,2} = Y_2\left(\frac{Y_1 + 1}{Y_1 Y_2}\right) = \left(Y_{1,1} + 1\right) = \left(\frac{1}{Y_1} + 1\right)$$

$$Y_{1,0}\, Y_{1,2} = Y_1\left(\frac{Y_2}{Y_1 Y_2 + Y_1 + 1}\right) = \left(Y_{2,1}^{-1} + 1\right)^{-1} = \left[\frac{Y_1 + 1}{Y_1 Y_2} + 1\right]^{-1}$$

$$Y_{2,1}\, Y_{2,3} = \frac{Y_1 Y_2}{Y_1 + 1} \cdot \frac{1}{Y_1 (Y_2 + 1)} = \left(Y_{1,2}^{-1} + 1\right)^{-1} = \left[\frac{Y_1 Y_2 + Y_1 + 1}{Y_2} + 1\right]^{-1}$$