Discrete Integrable Systems and Cluster Algebras

At the end of last week, we mentioned Zamolodchikov periodicity related to Dynkin Diagrams.

We talk about a more general case today corresponding to the "product" of two Dynkin Diagrams.

For this case, periodicity proven by B. Keller in 2013 using categorification (Published in Annals of Mathematics)

Let \( \Delta \) and \( \Delta' \) be two Dynkin diagrams with vertex sets \( I \) and \( I' \) both oriented in a bipartite i.e. alternating fashion.

**E.g.,** \( A_4 \) \[\begin{array}{l}
A_4 \ \bullet & \rightarrow & \circ & \leftarrow & \bullet & \rightarrow & \circ \\
end{array}\]

and \( D_5 \) \[\begin{array}{l}
D_5 \ \bullet & \rightarrow & \circ & \leftarrow & \bullet & \rightarrow & \circ \\
end{array}\]

We define the quiver associated to \( \Delta \times \Delta' \) by arranging \( \Delta \) horizontally and \( \Delta' \) vertically, and making a new graph on vertices \( (i, j) \in I \times I' \).

We color vertex \( (i, j) \) as \( \bullet \) if \( i \in \Delta \) and \( j \in \Delta' \) have the same color, and as \( \circ \) if \( i \in \Delta \) and \( j \in \Delta' \) have opposite colors.
we orient arrows of quiver associated to $\triangle \times \triangle'$ by $\bullet \rightarrow \circ$ in $\triangle$ and $\circ \rightarrow \bullet$ in $\triangle'$.

E.g., $A_4 \times D_5$

\[ \begin{array}{ccc}
\circ & \rightarrow & \circ \\
\circ & \rightarrow & \circ \\
\circ & \rightarrow & \circ \\
\circ & \rightarrow & \circ \\
\circ & \rightarrow & \circ \\
\end{array} \]

**Def:** Cartan matrices associated to $\triangle, \triangle'$ come from root systems/reflection group theory, essentially symmetrized version of each matrices assoc. to quiver $\triangle, \triangle'$.

Can define Cartan matrices for types $B_n, C_n, F_4, G_2$ also where skew-symmetrizable matrix but not a quiver.

E.g, For $\triangle$ type $A_5$, Cartan matrix =

\[
\begin{bmatrix}
2 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

always have $+2$'s on diagonal

non-positive off-diagonal

E.g, $\triangle$ type $B_5$,

\[
\begin{bmatrix}
2 & -1 & 0 & 0 & 0 \\
-2 & 2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
\end{bmatrix}
\]

$\triangle$ type $C_5$

\[
\begin{bmatrix}
2 & -2 & 0 & 0 & 0 \\
-2 & 2 & -2 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
\end{bmatrix}
\]

$\triangle$ type $G_2$

\[
\begin{bmatrix}
2 & -1 \\
-3 & 2 \\
\end{bmatrix}
\]
Y-system associated to \((\triangle \times \triangle')\) defined as

\[
\begin{align*}
Y_{i,j} & \cdot (c_{i,j}) \in \Delta \times \Delta', \ t \in \mathbb{Z}
\end{align*}
\]

satisfying

\[
Y_{i,j}^{t+1} \cdot Y_{i,j}^{t-1} = \prod_{j \in J} \left(1 + Y_{i,j}^{t-1}\right) - \alpha_{i,j}^{-1}
\]

For \(t\) odd (resp. even)

if \((c_{i,j} = 0)\) (resp. \(\circ\))

where \(\alpha_{i,j} = i,j\) entry of Cartan matrix for \(\triangle\)

\(\alpha_{i,j}' = i,j'\) entry of \(\triangle'\)

(exponents are nonnegative since off-diagonal entries of Cartan matrices are nonpositive.

Let \(h\) (resp. \(h'\)) be the Coxeter number associated to \(\triangle\) (resp. \(\triangle'\)) defined as

the period applying a certain cyclic transformation (Coxeter transformations) to the root system.

E.g., type \(A_n\) (corresponds to symm. group \(S_{n+1}\))

let \(s_i = \text{transposition} (i,i+1)\) for \(1 \leq i \leq n\)

let \(c = s_1 s_2 s_3 \cdots s_{2m} s_{2m+1} s_2 s_4 \cdots s_{2m} \) if \(m = \lfloor \frac{n}{2} \rfloor\)

\(c\) acts on a two-dim plane, has order \(n+1 = h\).
Thm (Keller) [Periodicity Conjecture]

\[ Y_{i,j,t+2(h+h')} = Y_{i,j,t} \quad \forall \, c \in \mathbb{Z} \]

History: Zamolodchikov conjectured \( \approx 1991 \)
Thermodynamic Bethe Ansatz] For \( \Delta \times A \) case w/ \( \Delta \) simply-laced (ADE).

This was the case discussed the end of last week.
Periodicity = \( 2(h+2) \) in this case (since \( h'=1 \)) for \( \Delta' = A_1 \).

For \( \Delta = A_n \), \( \Delta' = A_1 \), first proven w/ explicit solutions in Frenkel-Szenes \( \approx 1995 \).

Independently by Gliozzi-Takeo (using volumes of 3-folds and triangulations) \( \approx 1996 \).

\( \Delta \times A_m \) for any \( \Delta \) (not necr. simply-laced) using cluster algebras \( \approx 2001/2002 \) by Fomin-Zelevinsky.

\( A_n \times A_m \) case by Volkov \( \approx 2006 \) (explicit formulas using cross-ratios).

E.g., Looks like (in \( A_n \times A_m \) case)

\[ Y_{i,j,t+1} Y_{i,j,t-1} = \frac{(1 + Y_{i,j,t})(1 + Y_{i,j,t+1})}{(1 + Y_{i,j,t}) (1 + Y_{i,j,t+1})} \]

For \( 1 \leq i \leq n \), \( 1 \leq j \leq m \)

Setting boundary conditions \( i = 0, n+1 \) & \( j = 0, m+1 \).
Relation with cluster algebras: Define coloring of vertices of quiver assoc. to \( \Delta \times \Delta' \) by 0 \& \( \bullet \) as above.

\[ \tau_+ = \text{Mutate at all } \bullet \]

\[ \tau_- = \text{Mutate at all } \circ \]

We think of \( \tau_+ \) (resp. \( \tau_- \)) as multiple mutations simultaneously (equivalently sequentially by any order).

Since no vertices of same color border each other, such mutations commute.

Result after \( \tau_+ \) all arrows reversed.

Following by \( \tau_- \) reverses all arrow back to the original.

We define \( Y_{ij,j'}^{t+1} \) as the \( Y \)-system value at \( (i,j) \).

After applying \( \tau_-/\tau_+ \) let \( Y_{ij,j'}^{t+1} \) be the updated value.

Here \( Y_{ij,j'}^{t+1} \) is value after \( \tau_+ \tau_-/\tau_- \tau_+ \).

Same quiver and thus can iterate or run backwards to get values for all \( t \in \mathbb{Z} \).
By definition of $Y$-seed mutation

After $\tau_+$ all vertices are $Y_{i,j,t-1}$

their neighbors $(i,j')$ vertices updated to

$Y_{i,j',t-1} \rightarrow Y_{i,j',t-1} \times (1 + Y_{i,j',t-1})$ if $(i,j') (c,i')$

mutation at a single vertex

$Y_{i,j',t-1} \times (1 + Y_{i,j',t-1})$ if $(c,i') (i,j')$

Taking the full composite mutation sequence

$Y_{i,j',t-1} \xrightarrow{\tau_+} Y_{i,j',t-1} \prod (1 + Y_{i,j',t-1})$

We then apply $\tau_-$

all vertices are inverted

$\Rightarrow Y_{i,j',t+1} = Y_{i,j',t-1} \prod (1 + Y_{i,j',t-1})$

$Y_{i,j',t} \rightarrow Y_{i,j',t-1}$
Lastly, we note that \( Y_{i,j,t} = Y_{i,j,t-1} \) for \((i,j)^t\).

and so rewrite For \((i,j)^t'\) = 0

\[
Y_{i,j,t+1} \times Y_{i,j,t-1} = \prod \left( 1 + Y_{i,j,t} \right)
\]

as desired

\[
Y_{i,j,t}^{-1}
\]

The proof for \( Y_{i,j,t+1} \)'s where \((i,j)^t\) = 0

is analogous except we need \( t \) even rather than odd this time

(so that we mutate \( \tau - \) then \( \tau + \) instead).

Note that even in the \( A_2 \times A_2 \) case

the \( Y \)-system already non-trivial.

\( h = h' = 3 \) for this case. By the Thm

\[
\Rightarrow \text{ period for this } Y \text{-system is } 2(h+h') = 12.
\]

we illustrate the first six steps

which corresponds to a 180°-rotation.
Sketch of Proofs

For original \( \triangle \times A_1 \) (Zamolodchikov) case,

there exists a root system associated to \( \triangle \)

(e.g., \( \{e_i-e_j \mid 1 \leq i,j \leq n+1 \} \) for \( A_n \) case)

Letting \( \alpha_1 = e_1 - e_2 \), \( \alpha_2 = e_2 - e_3 \), ..., \( \alpha_n = e_n - e_{n+1} \)

we can rewrite \( e_i - e_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1} \).

Each mutation \( M_i \) in sequence \( \tau_+ \circ \tau_- \) looks locally like \( i \leftrightarrow \overline{i} \) after mutation.

\[ x_i \overline{t-1} \cdot x_i \overline{t+1} = x_i \overline{t-1} \cdot x_i \overline{t+1} + 1 \]

We let initial cluster be

\[ \{x_1, x_2, ..., x_n\} = \{x_{1,0}, x_{2,1}, x_{3,0}, ..., x_{m,n}, x_{n,n+1}\} \]

After \( (\tau_+ \circ \tau_-)^k \)

\[ \{x_{1,2k}, x_{2,2k+1}, x_{3,2k}, ..., x_{m,2k+1}, x_{n,2k}\} \]

Initial cluster \( \leftrightarrow \) roots \( \{-\alpha_1, -\alpha_2, -\alpha_3, ..., -\alpha_{n+1}, -\alpha_n\} \)

For non-initial \( x_{i,j} \) let \( x_{i,j} = \frac{P_{i,j}(x_1, ..., x_n)}{x_1^{d_1} ... x_n^{d_n}} \).
Let $s_c = \text{root system reflection (of } \alpha_i \leftrightarrow -\alpha_i)$

$$s_c(\alpha_j) = \alpha_j - \frac{2 \langle \alpha_i, \alpha_j \rangle \alpha_i}{\langle \alpha_i, \alpha_i \rangle}$$

Rem: $q_{ij} = \frac{2 \langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \frac{11\alpha_j}{11\alpha_i} \cos \theta$ are the values of the associated Cartan matrix.

Can be shown: If $t = 2k_j ~ \& ~ X_{ij}$ as above then

$$d_1 \alpha_1 + d_2 \alpha_2 + \ldots + d_n \alpha_n = \left( s_1 s_2 \ldots s_{m+1} s_2 s_4 \ldots s_m \right) \alpha_j$$

Here periodicity of sequence of Y-system follows by periodicity of cluster variables, which follows periodicity/finiteness of the root system associated to $\Delta_0$.

**Example (A_2) $\cdot \rightarrow \mathbb{R}^2$ w/ Cartan matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$**

We can embed the associated root system in $\mathbb{R}^2$ as $\alpha_2 \rightarrow \alpha_1$.

Thus $-\alpha_1 \rightarrow \alpha_1, \alpha_2 \rightarrow \alpha_1 + \alpha_2, \alpha_2 \rightarrow -\alpha_2$.

For more complicated examples we use piecewise-linear analogue:

$$\sigma_c(\alpha) = \begin{cases} s_c(\alpha) \text{ unless } \alpha = -\alpha_j \text{ for } j \neq c \\ \alpha \text{ if } \alpha = \alpha_j \text{ for } j \neq c \end{cases}$$

$G_2$-orbits (using $\sigma_c(\alpha)$'s) yield all pos roots & periodicity of cl. vars & Y-systems (from negative simples).
For full $\Delta \times \Delta'$ case, Keller uses category theory to mimic root system combinatorics.

Auslander-Reiten translation $\tau$ on bounded derived category $\mathcal{D}^b(k\Lambda)$ of modules over the path algebra $k\Lambda$ plays the role of $\tau_+ \tau_- = C$ (Coxeter elt) in refl. gp.

Thms of Gabriel & Happel $\Rightarrow$ periodicity of AR tran., acting of Grothendieck group $K_0(D^b)$

Another nuance: since $\Delta \times \Delta'$ contains cycles

use quivers w/potentials to deal w/ 2-cycles categorically (not just simply reversing sources to sinks throughout)