Beyond Aztec Castles: Toric Cascades in the $dP3$ Quiver

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http://math.umn.edu/~musiker/Fargo.pdf
Outline.

1. Introduction to Cluster Algebras
2. What is a Brane Tiling
3. The Del Pezzo 3 Quiver and Lattice
4. Aztec Castles and Beyond (work of Leoni-M-Neel-Turner and Lai-M)

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What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ is a subalgebra of $k(x_1, \ldots, x_n)$ constructed cluster by cluster by certain exchange relations.

Generators:

Specify an initial finite set of them, a Cluster, $\{x_1, x_2, \ldots, x_n\}$.

Construct the rest via Binomial Exchange Relations:

$$x_{\alpha}x_{\alpha}' = \prod x_{\gamma_i}^{d_i^+} + \prod x_{\gamma_i}^{d_i^-}.$$

The set of all such generators are known as Cluster Variables, and the initial pattern of exchange relations determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.
Quiver Mutation

We focus on cluster algebras whose initial pattern of exchange relations is determined by a quiver, i.e. a directed graph

\[
\begin{array}{ccc}
1 & \xrightarrow{} & 2 \\
\cdot & & \cdot \leftarrow \\
& 3 & \\
\end{array}
\]

\{x_1, x_2, x_3\}

\[
x_j x_j' = \prod_{i \to j \in Q} x_i + \prod_{j \to i \in Q} x_i,
\]

\[
\text{i.e. } x_j x_j' = \prod_{i \to j \in Q} x_i^{d_i^+} + \prod_{j \to i \in Q} x_i^{d_i^-}
\]

where \(d_i^+\) is the number of arrows from vertex \(i\) to \(j\) and \(d_j^-\) is the number of arrows from vertex \(j\) to \(i\).

Example: Mutating at vertex 2 yields \(x_2' x_2 = x_1 + x_3\)

\[
\begin{array}{ccc}
1 & \xrightarrow{} & 2 \\
\cdot & & \cdot \leftarrow \\
& 3 & \\
\end{array}
\]

\{x_1, \frac{x_1 + x_3}{x_2}, x_3\}

Observe: we also mutate the quiver \(Q\) and obtain a new exchange pattern.
Quiver Mutation (at vertex $j$)

1st) Add an edge $i \to k$ for every 2-path $i \to j \to k$ in $Q$, the original quiver.

2nd) Reverse all arrows, i.e. directed edges, incident to vertex $j$.

3rd) Lastly, we erase all 2-cycles (that have been created by steps 1 and 2), and denote the resulting quiver as $\mu_j(Q)$. 

\[
\begin{align*}
\{x_1, x_2, x_3\} \\
\{x_1, \frac{x_1 + x_3}{x_2}, x_3\}
\end{align*}
\]
Let $\mathcal{A}$ be the cluster algebra defined by the initial cluster \{\(x_1, x_2, x_3\)\} and the initial exchange pattern

\[x_1x'_1 = 1 + x_2, \quad x_2x'_2 = x_1x_3 + 1, \quad x_3x'_3 = 1 + x_2.\]

corresponding to the quiver

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\bullet & & \bullet \\
& \rightarrow & \\
& & 3
\end{array}
\]

$\mathcal{A}$ is of finite type, type $A_3$, generated by the cluster variables

\[
\left\{ x_1, x_2, x_3, \frac{1 + x_2}{x_1}, \frac{x_1x_3 + 1}{x_2}, \frac{1 + x_2}{x_3}, \frac{x_1x_3 + 1 + x_2}{x_1x_2}, \frac{x_1x_3 + 1 + x_2}{x_2x_3}, \frac{x_1x_3 + 1 + x_2 + x_2 + x_2^2}{x_1x_2x_3} \right\}.
\]
Kronecker Quiver, otherwise known as (Affine Type, of Type $\tilde{A}_1$) or corresponding to an **annulus with two marked points.**

\[ x_3 = \frac{x_2^2 + 1}{x_1}. \]

\[ \bullet_1 \implies \bullet_2 \quad \text{yields} \quad x_n x_{n-2} = x_{n-1}^2 + 1. \]
Kronecker Quiver, otherwise known as (Affine Type, of Type $\tilde{A}_1$) or corresponding to an annulus with two marked points.

\begin{align*}
  \bullet_1 & \implies \bullet_2 \quad \text{yields} \quad x_n x_{n-2} = x_{n-1}^2 + 1. \\
  x_3 &= \frac{x_2^2 + 1}{x_1}. \quad x_4 = \frac{x_3^2 + 1}{x_2} = \frac{x_2^4 + 2x_2^2 + 1 + x_1^2}{x_1^2 x_2}.
\end{align*}
Second Example of a Cluster Algebra

**Kronecker Quiver**, otherwise known as (Affine Type, of Type \( \tilde{A}_1 \)) or corresponding to an annulus with two marked points.

\[ \bullet_1 \implies \bullet_2 \] yields \[ x_n x_{n-2} = x_{n-1}^2 + 1. \]

\[ x_3 = \frac{x_2^2 + 1}{x_1}, \quad x_4 = \frac{x_3^2 + 1}{x_2} = \frac{x_2^4 + 2x_2^2 + 1 + x_1^2}{x_1^2 x_2}. \]

\[ x_5 = \frac{x_4^2 + 1}{x_3} = \frac{x_2^6 + 3x_2^4 + 3x_2^2 + 1 + x_1^4 + 2x_1^2 + 2x_1^2 x_2^2}{x_1^3 x_2}, \ldots \]
Second Example of a Cluster Algebra

Kronecker Quiver, otherwise known as (Affine Type, of Type $\tilde{\Lambda}_1$) or corresponding to an annulus with two marked points.

\[ \bullet_1 \implies \bullet_2 \] yields \[ x_n x_{n-2} = x_{n-1}^2 + 1. \]

\[ x_3 = \frac{x_2^2 + 1}{x_1}, \quad x_4 = \frac{x_3^2 + 1}{x_2} = \frac{x_2^4 + 2x_2^2 + 1 + x_1^2}{x_1^2 x_2}. \]

\[ x_5 = \frac{x_4^2 + 1}{x_3} = \frac{x_2^6 + 3x_2^4 + 3x_2^2 + 1 + x_1^4 + 2x_1^2 + 2x_1^2 x_2^2}{x_1^3 x_2}, \ldots \]

If we let $x_1 = x_2 = 1$, we obtain \( \{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\} \).
Second Example of a Cluster Algebra

**Kronecker Quiver**, otherwise known as (Affine Type, of Type \(\widetilde{A}_1\)) or corresponding to an **annulus with two marked points**.

\[ \bullet_1 \implies \bullet_2 \quad \text{yields} \quad x_n x_{n-2} = x_{n-1}^2 + 1. \]

\[ x_3 = \frac{x_2^2 + 1}{x_1}, \quad x_4 = \frac{x_3^2 + 1}{x_2} = \frac{x_2^2 + 2x_2^2 + 1 + x_1^2}{x_1^2 x_2}. \]

\[ x_5 = \frac{x_4^2 + 1}{x_3} = \frac{x_2^6 + 3x_2^4 + 3x_2^2 + 1 + x_1^4 + 2x_1^2 + 2x_1^2 x_2^2}{x_1^3 x_2}, \quad \ldots \]

If we let \(x_1 = x_2 = 1\), we obtain \(\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}\).

The next number in the sequence is \(x_7 = \frac{34^2 + 1}{13} = \).
Second Example of a Cluster Algebra

**Kronecker Quiver**, otherwise known as (Affine Type, of Type $\tilde{A}_1$) or corresponding to an annulus with two marked points.

\[ \bullet_1 \implies \bullet_2 \] yields \[ x_n x_{n-2} = x_{n-1}^2 + 1. \]

\[
\begin{align*}
x_3 &= \frac{x_2^2 + 1}{x_1} \quad x_4 &= \frac{x_3^2 + 1}{x_2} = \frac{x_2^4 + 2x_2^2 + 1 + x_1^2}{x_1^2 x_2} \\
x_5 &= \frac{x_4^2 + 1}{x_3^2} = \frac{x_2^6 + 3x_2^4 + 3x_2^2 + 1 + x_1^4 + 2x_1^2 + 2x_1^2 x_2^2}{x_1^3 x_2^2}, \ldots
\end{align*}
\]

If we let $x_1 = x_2 = 1$, we obtain \( \{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\} \).

The next number in the sequence is $x_7 = \frac{34^2 + 1}{13} = \frac{1157}{13} =$
Second Example of a Cluster Algebra

**Kronecker Quiver**, otherwise known as (Affine Type, of Type $\tilde{A}_1$) or corresponding to an annulus with two marked points.

\[
\bullet_1 \implies \bullet_2 \quad \text{yields} \quad x_n x_{n-2} = x_{n-1}^2 + 1.
\]

\[
x_3 = \frac{x_2^2 + 1}{x_1}, \quad x_4 = \frac{x_3^2 + 1}{x_2} = \frac{x_2^4 + 2x_2^2 + 1 + x_1^2}{x_1^2 x_2},
\]

\[
x_5 = \frac{x_4^2 + 1}{x_3} = \frac{x_2^6 + 3x_2^4 + 3x_2^2 + 1 + x_1^4 + 2x_1^2 + 2x_1^2 x_2}{x_1^3 x_2}, \quad \ldots
\]

If we let $x_1 = x_2 = 1$, we obtain $\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}$.

The next number in the sequence is $x_7 = \frac{34^2 + 1}{13} = \frac{1157}{13} = 89$, an integer!

This is an example of a cluster algebra of finite mutation type.
A cluster algebra is of **finite type** if the number of cluster variables and the number of quivers reachable via mutations is **finite**.

A cluster algebra is of **finite mutation type** if the number of quivers reachable via mutations is finite (but the number of cluster variables could be infinite).

A cluster algebra is of **infinite mutation type** if both the number of cluster variables and the number of quivers reachable via mutations is **infinite**.

Most cluster algebras of finite mutation type come from a surface (e.g. Kronecker quiver comes from an annulus).

We now shift our focus to cluster algebras of **infinite mutation type**.
Consider the quiver $Q$ (on the left below). Instead of all cluster variables, we focus on those obtained by mutating $1, 2, 3, 4, 1, 2, \ldots$ periodically:

$$
\begin{array}{c}
1 & \rightarrow & 3 \\
\downarrow & & \downarrow \\
4 & \rightarrow & 2 \\
\end{array}
\rightarrow
\begin{array}{c}
1 & \rightarrow & 3 \\
\downarrow & & \downarrow \\
4 & \rightarrow & 2 \\
\end{array}
\rightarrow
\begin{array}{c}
1 & \rightarrow & 3 \\
\downarrow & & \downarrow \\
4 & \rightarrow & 2 \\
\end{array}
\rightarrow \ldots
$$
Consider the quiver $Q$ (on the left below). Instead of all cluster variables, we focus on those obtained by mutating $1, 2, 3, 4, 1, 2, \ldots$ periodically:

Yields a sequence of cluster variables, with initial cluster variables $x_1, x_2, x_3, x_4$, with $x_{n+4}$ denoting the $n$th new cluster variable obtained by this mutation sequence $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, \ldots\}$.

Because of the periodicity, it follows that the $x_n$’s satisfy the recurrences

$$x_n x_{n-4} = \begin{cases} x_{n-1}^2 + x_{n-2}^2 & \text{when } n \text{ is odd, and} \\ x_{n-2}^2 + x_{n-3}^2 & \text{when } n \text{ is even.} \end{cases}$$

For example, $x_5 = \frac{x_3 + x_4^2}{x_1}$, $x_6 = \frac{x_3 + x_4^2}{x_2}$, $x_7 = \frac{x_5 + x_6^2}{x_3}$, and $x_8 = \frac{x_5^2 + x_6^2}{x_4}$. 
Let $Q = \begin{array}{ccc}
1 & 3 \\
4 & 2 
\end{array}$, and mutate periodically at $1, 2, 3, 4, 1, 2, 3, 4, \ldots$.

$x_n x_{n-4} = \begin{cases} 
x^2_{n-1} + x^2_{n-2} & \text{when } n \text{ is odd, and} \\
x^2_{n-2} + x^2_{n-3} & \text{when } n \text{ is even.}
\end{cases}$

By letting $x_1 = x_2$ and $x_3 = x_4$, we get $x_{2n+1} = x_{2n}$ for all $n$.

Letting $\{ T_n \}$ be the sequence $\{x_{2n}\}_{n \in \mathbb{Z}}$, we obtain a single recurrence.

$$T_n T_{n-2} = 2 T_{n-1}^2.$$ 

If $T_1 = T_2 = 1$, $\{ T_n \} = \{1, 1, 2, 8, 64, 1024, 32768, \ldots \} = \left\{ 2^{(n-1)(n-2)/2} \right\}$.

For $n \geq 3$, $T_n = \#$ (perfect matchings of the $(n-2)$nd Aztec Diamond).
Let $Q = \begin{bmatrix} 2 & 4 & 2 & 4 & 2 & 4 & 2 \\ 3 & 1 & 3 & 1 & 3 & 1 & 3 \\ 2 & 4 & 2 & 4 & 2 & 4 & 2 \\ 3 & 1 & 3 & 1 & 3 & 1 & 3 \\ 2 & 4 & 2 & 4 & 2 & 4 & 2 \\ 3 & 1 & 3 & 1 & 3 & 1 & 3 \\ 2 & 4 & 2 & 4 & 2 & 4 & 2 \end{bmatrix}$, and mutate periodically at 1, 2, 3, 4, 1, 2, 3, 4, \ldots.

$x_5 = \frac{x_3^2 + x_4^2}{x_1}$, $x_6 = \frac{x_3^2 + x_4^2}{x_2}$, $x_7 = \frac{(x_3^2 + x_4^2)(x_2^2 + x_1^2)}{x_1^2 x_2^2 x_3}$, and $x_8 = \frac{(x_3^2 + x_4^2)^2 (x_2^2 + x_1^2)}{x_1^2 x_2^2 x_4}$. 

Lai-Musiker (University of Minnesota) Beyond Aztec Castles

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The Del Pezzo 3 Quiver and Aztec Dragons

Mutating 1, 2, 3, 4, 5, 6, 1, 2, 3, ...

Introduced by Jim Propp, Ben Wieland, and Mihai Ciucu. Studied further by Cottrell-Young.

\[ x_{2n+7}x_{2n+1} = x_{2n+3}x_{2n+5} + x_{2n+4}x_{2n+6} \]

and

\[ x_{2n+8}x_{2n+2} = x_{2n+3}x_{2n+5} + x_{2n+4}x_{2n+6}. \]
**Toric Mutations and Toric Phases of \( dP_3 \)**

**Toric mutations** take place at vertices with in-degree and out-degree 2.

Starting with any of these four models of the \( dP_3 \) quiver, any sequence of toric mutations yields a quiver that is graph isomorphic to one of these.

Figure 20 of Eager-Franco (Incidences between these Models):
Example from S. Zhang (2012 REU): Periodic mutation 1, 2, 3, 4, 5, 6, 1, 2, ... yields partition functions for Aztec Dragons (as studied by Ciucu, Cottrell-Young, and Propp) under appropriate weighted enumeration of perfect matchings.
Example from M. Leoni, S. Neel, and P. Turner (2013 REU):
Mutations at antipodal vertices of $dP_3$ quiver yield $\tau$-mutation sequences. Resulting Laurent polynomials correspond to Aztec Castles under appropriate weighted enumeration of perfect matchings.

E.g. $1, 2, 3, 4, 1, 2, 5, 6$ yields cluster variable

\[
\begin{align*}
(x_1x_2^2x_3^3x_5^4 + x_3x_4^2x_5^4 + 2x_1^2x_2x_3^3x_5^4x_6 + 4x_1x_2^2x_3x_4x_5^3x_6 + 2x_2^3x_3^2x_5^3x_6 + x_1^3x_3^3x_5^2x_6 \\
+ 5x_1^2x_2x_3x_4x_5^5x_6 + 5x_1^3x_2x_3x_4x_5^5x_6 + x_2^3x_4^2x_5^2x_6 + 2x_2^3x_3^2x_4x_5^6x_6 + x_1^3x_3^2x_4^3x_5x_6 \\
+ 2x_1^2x_2x_4x_5x_6^3 + x_1^3x_3x_4^2x_6^4 + x_1^2x_2x_4x_6^4) / x_1^2x_2^2x_3^2x_4x_6 &= \frac{(x_1x_3 + x_2x_4)(x_4x_6 + x_3x_5)^2(x_1x_6 + x_2x_5)^2}{x_1^2x_2^2x_3^2x_4^2x_6}
\end{align*}
\]
Theorem 1 [Lai-M 2015] Starting from the initial cluster \( \{x_1, x_2, \ldots, x_6\} \), the set of cluster variables reachable via toric mutations can be parameterized by \( \mathbb{Z}^3 \).

Under this correspondence, the initial cluster bijects to

\[
[(0, -1, 1), (0, -1, 0), (-1, 0, 0), (-1, 0, 1), (0, 0, 1), (0, 0, 0)]
\]

and toric mutations transform the six-tuple in \( \mathbb{Z}^3 \) as we will illustrate.

Up to symmetry, enough to consider \( \mu_1 \mu_2, \mu_1 \mu_4 \mu_1 \mu_5 \mu_1 \), and \( \mu_1 \mu_4 \mu_3 \).
Mutating Model I to Model II and back to Model I

By applying $\mu_1 \circ \mu_2$, $\mu_3 \circ \mu_4$, or $\mu_5 \circ \mu_6$, we mutate the quiver (up to graph isomorphism):

Corresponding action in $\mathbb{Z}^3$ (on triangular prisms):
Illustrating the mutation sequence $\mu_1\mu_4\mu_1\mu_5\mu_1$
Illustrating the mutation sequence $\mu_1\mu_4\mu_3$
Segway: Algebraic Formula for Toric Cluster Variables

Let \( A = \frac{x_3x_5 + x_4x_6}{x_1x_2} \), \( B = \frac{x_1x_6 + x_2x_5}{x_3x_4} \), \( C = \frac{x_1x_3 + x_2x_4}{x_5x_6} \),

\[ D = \frac{x_1x_3x_6 + x_2x_3x_5 + x_2x_4x_6}{x_1x_4x_5} \]

and \( E = \frac{x_2x_4x_5 + x_1x_3x_5 + x_1x_4x_6}{x_2x_3x_6} \).

Let \( z_{i,j,k}^r \) be the cluster variable corresponding to \((i, j, k) \in \mathbb{Z}^3\).

**Theorem 2 [Lai-M 2015]** (Extension of [LMNT 2013] and [Lai 2014]):

\[ z_{i,j,k}^r = x_r \quad A^{\left(\frac{i^2 + ij + j^2 + 1}{3} + i + 2j\right)} \quad B^{\left(\frac{i^2 + ij + j^2 + 1}{3} + 2i + j\right)} \quad C^{\left(\frac{i^2 + ij + j^2 + 1}{3}\right)} \quad D^{\left(\frac{(k-1)^2}{4}\right)} \quad E^{\left(\frac{k^2}{4}\right)} \]

where, working modulo 6, we have (cyclically around the \(dP_3\) Quiver)

- \( r = 6 \) if \( 2(i - j) + 3k \equiv 0 \),
- \( r = 4 \) if \( 2(i - j) + 3k \equiv 1 \),
- \( r = 2 \) if \( 2(i - j) + 3k \equiv 2 \),
- \( r = 5 \) if \( 2(i - j) + 3k \equiv 3 \),
- \( r = 3 \) if \( 2(i - j) + 3k \equiv 4 \),
- \( r = 1 \) if \( 2(i - j) + 3k \equiv 5 \).

i.e. we determine \( x_r \) by looking at \((i - j)\) modulo 3 and \(k\) modulo 2.
Segway: Algebraic Formula for Toric Cluster Variables

\[ Z_{i-1}^{j+2,k} z_{j,k+1} Z_i = (R4) Z_{i-1}^{j+1,k} z_{j+1,k+1} + Z_{i-1}^{j+1,k+1} z_{j+1,k} \]

\[ Z_{i+1}^{j+1,k+1} z_{j+1,k-1} Z_i = (R1) Z_{i-1}^{j+2,k} z_{j,k} + Z_{i-1}^{j+1,k} z_{j+1,k+1} \]

\[ Z_{i+1}^{j+2,k} z_{j,k} Z_i = (R2) Z_{i-1}^{j+1,k-1} z_{j+1,k+1} + (z_{j+1,k})^2 \]

\[ Z_{i-1}^{j+1,k+1} z_{j,k-1} Z_i = (R1) Z_{i+1}^{j,k} z_{j+1,k-1} + Z_{i+1}^{j,k} z_{j+1,k+1} \]

\[ Z_{i+1}^{j,k} z_{j+1,k-1} Z_i = (R4) Z_{i-1}^{j,k} z_{j+1,k} + Z_{i-1}^{j-1,k-1} z_{j+1,k+1} \]

(I) \leftrightarrow (II) \leftrightarrow (III) \leftrightarrow (III) \leftrightarrow (II) \leftrightarrow (I)

From work of Lai "A generalization of Aztec Dragons": Unweighted versions of these recurrences called type \((R1), (R2), or (R4)\) recurrences.
Introducing Contours on the del Pezzo 3 Lattice

We wish to understand combinatorial interpretations for more general toric mutation sequences, not necessarily periodic or coming from mutating at antipodes.

To this end, we cut out subgraphs of the $dP_3$ lattice by using six-sided contours indexed as $(a, b, c, d, e, f)$ with $a, b, c, d, e, f \in \mathbb{Z}$.
Sign determines direction & Magnitude determines length

(1) \( G(3, 2, 4, 2, 3, 3) \), (2) \( G(3, -4, 2, 2, -3, 1) \), (3) \( G(3, -4, 4, 0, -1, 1) \),
(4) \( G(5, -6, 3, 0, -1, -2) \), (5) \( G(5, -6, 2, 2, -3, -1) \), (6) \( G(2, 1, 1, -3, 6, -4) \)
\[ D_{n+1/2} = G(n+1, -n, -1, n+2, -n-1, 0), \quad D_n = G(n+1, -n-1, 1, n, -n, 0). \]
Turning Subgraphs into Laurent Polynomials

\[ G \rightarrow cm(G) \sum_{M = \text{a perfect matching of } G} x(M), \text{ where} \]

\[ x(M) = \prod_{\text{edge } e \in M} \frac{1}{x_i x_j} \text{ (for edge } e \text{ straddling faces } i \text{ and } j), \]

\[ cm(G) = \text{the covering monomial of the graph } G_n \text{ (which records what face labels are contained in } G \text{ and along its boundary).} \]

**Remark:** This is a reformulation of weighting schemes appearing in works such as Speyer ("Perfect Matchings and the Octahedron Recurrence"), Goncharov-Kenyon ("Dimers and cluster integrable systems"), and Di Francesco ("T-systems, networks and dimers").

**Alternative definition of cm(G):** We record all face labels inside contour and then divide by face labels straddling dangling edges.
Initial cluster \( \{x_1, x_2, \ldots, x_6\} \) in terms of contours

Consider the following six special contours

\[
C_1 = (0, 0, 1, -1, 1, 0), \quad C_2 = (-1, 1, 0, 0, 0, 1), \\
C_3 = (0, 1, -1, 1, 0, 0), \quad C_4 = (1, 0, 0, 0, 1, -1), \\
C_5 = (1, -1, 1, 0, 0, 0), \quad C_6 = (0, 0, 0, 1, -1, 1).
\]

Applying our general algorithm, \( \mathcal{G}(C_i) \)'s correspond to graphs consisting of a single edge and a triangle of faces.
Initial cluster \( \{x_1, x_2, \ldots, x_6\} \) in terms of contours

Consider the following six special contours

\[
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C_3 = (0, 1, -1, 1, 0, 0), \quad C_4 = (1, 0, 0, 0, 1, -1), \\
C_5 = (1, -1, 1, 0, 0, 0), \quad C_6 = (0, 0, 0, 1, -1, 1).
\]

Applying our general algorithm, \( \mathcal{G}(C_i) \)'s correspond to graphs consisting of a single edge and a triangle of faces.

Using \( G \rightarrow cm(G) \sum_M = a \) perfect matching of \( G \times (M) \), we see

\[
\text{cm}(\mathcal{G}(C_1)) = x_1x_4x_5 \quad \text{and} \quad x(M) = \frac{1}{x_4x_5}, \quad \text{hence} \quad G \rightarrow \frac{x_1x_4x_5}{x_4x_5} = x_1
\]

Similar calculations show \( \mathcal{G}(C_i) \leftrightarrow x_i \) for \( i \in \{1, 2, \ldots, 6\} \).
Theorem 3 [Lai-M 2015]

**Theorem (Reformulation of [Leoni-M-Neel-Turner 2014]):** Let $Z^S = [z_1, z_2, \ldots, z_6]$ be the cluster obtained after applying a toric mutation sequence $S$ to the initial cluster $\{x_1, x_2, \ldots, x_6\}$.

Let $w(G) = cm(G) \sum_M$ a perfect matching of $G \times (M)$.

Let $G(C_i)$ be the subgraph cut out by the contour $C_i$.

Then $Z^S = [w(G(C_1^S)), w(G(C_2^S)), \ldots, w(G(C_6^S))]$ where $C_1^S, C_2^S, \ldots, C_6^S$ are defined as follows:

1) Start with the six-tuple $[(0, -1, 1), (0, -1, 0), (-1, 0, 0), (-1, 0, 1), (0, 0, 1), (0, 0, 0)]$ in $\mathbb{Z}^3$.

2) Toric Mutations transform this six-tuple as illustrated earlier.

3) Map from $\mathbb{Z}^3$ to $\mathbb{Z}^6$:

$$(i, j, k) \rightarrow (a, b, c, d, e, f) = (j + k, -i - j - k, i + k, j - k + 1, -i - j + k - 1, i - k + 1)$$

and use these six six-tuples to define the contours $C_1^S, C_2^S, \ldots, C_6^S$. 
Example 1: mutation sequence $\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6$

We start at the initial prism $[(0, -1), (0, 0), (0, 0), (0, 1), (0, 0), (0, 0)]$.

Applying the mutation sequence $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6$ corresponds to the walk

{$(0, -1), (0, 0), (0, 0)\} \rightarrow \{(-1, 1), (-1, 0), (0, 0)\} \rightarrow \{(-1, 1), (0, 1), (0, 0)\} \rightarrow \{(-1, 1), (0, 1), (-1, 2)\}$

Projecting to $\mathbb{Z}^2$ using $(i, j) \leftrightarrow (j, -i - j, i, j + 1, -i - j - 1, i + 1)$ and $(j + 1, -i - j - 1, i + 1, j, -i - j, i)$.

$C_1 = (0, 0, 1, -1, 1, 0), C_2 = (-1, 1, 0, 0, 0, 1), C_3 = (0, 1, -1, 1, 0, 0), C_4 = (1, 0, 0, 0, 1, -1), C_5 = (1, -1, 1, 0, 0, 0), C_6 = (0, 0, 0, 1, -1, 1)$. 
Example 1: mutation sequence $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6$

We start at the initial prism $[(0, -1, 1), (0, -1, 0), (-1, 0, 0), (-1, 0, 1), (0, 0, 1), (0, 0, 0)]$.

Applying the mutation sequence $\mu_1, \mu_2, \mu_3\mu_4\mu_5\mu_6$ corresponds to the walk

$\{(0, -1), (-1, 0), (0, 0)\} \rightarrow \{(-1, 1), (-1, 0), (0, 0)\} \rightarrow \{(-1, 1), (0, 1), (0, 0)\} \rightarrow \{(-1, 1), (0, 1), (-1, 2)\}$

Projecting to $\mathbb{Z}^2$ using $(i, j) \leftrightarrow (j, -i - j, i, j + 1, -i - j - 1, i + 1)$ and

$(j + 1, -i - j - 1, i + 1, j, -i - j, i)$.

$C'_1 = (2, -1, 0, 1, 0, -1), C'_2 = (1, 0, -1, 2, -1, 0), C_3 = (0, 1, -1, 1, 0, 0), C_4 = (1, 0, 0, 0, 1, -1), C_5 = (1, -1, 1, 0, 0, 0), C_6 = (0, 0, 0, 1, -1, 1)$. 
Example 1: mutation sequence $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6$

We start at the initial prism $[(0, -1, 1), (0, -1, 0), (-1, 0, 0), (-1, 0, 1), (0, 0, 1), (0, 0, 0)]$.
Applying the mutation sequence $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6$ corresponds to the walk

$$\{(0, -1), (-1, 0), (0, 0)\} \rightarrow \{(-1, 1), (-1, 0), (0, 0)\} \rightarrow \{(-1, 1), (0, 1), (0, 0)\} \rightarrow \{(-1, 1), (0, 1), (-1, 2)\}$$

Projecting to $\mathbb{Z}^2$ using $(i, j) \leftrightarrow (j, -i - j, i, j + 1, -i - j - 1, i + 1)$ and $(j + 1, -i - j - 1, i + 1, j, -i - j, i)$.

$C'_1 = (2, -1, 0, 1, 0, -1), C'_2 = (1, 0, -1, 2, -1, 0), C'_3 = (1, -1, 0, 2, -2, 1), C'_4 = (2, -2, 1, 1, -1, 0), C'_5 = (1, -1, 1, 0, 0, 0), C'_6 = (0, 0, 0, 1, -1, 1)$. 

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Example 1: mutation sequence $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6$

We start at the initial prism $[(0, 0), (0, -1), (0, 0), (-1, 0), (0, 0), (0, 0)]$.
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Projecting to $\mathbb{Z}^2$ using $(i, j) \leftrightarrow (j, -i - j, i, j + 1, -i - j - 1, i + 1)$ and

$(j + 1, -i - j - 1, i + 1, j, -i - j, i)$.

$C_1' = (2, -1, 0, 1, 0, -1)$, $C_2' = (1, 0, -1, 2, -1, 0)$, $C_3' = (1, -1, 0, 2, -2, 1)$,
$C_4' = (2, -2, 1, 1, -1, 0)$, $C_5' = (3, -2, 0, 2, -1, -1)$, $C_6' = (2, -1, -1, 3, -2, 0)$. 
Example 1: mutation sequence $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6$

$C'_1 = (2, -1, 0, 1, 0, -1), C'_2 = (1, 0, -1, 2, -1, 0), C'_3 = (1, -1, 0, 2, -2, 1), C'_4 = (2, -2, 1, 1, -1, 0), C'_5 = (3, -2, 0, 2, -1, -1), C'_6 = (2, -1, -1, 3, -2, 0).$
Example 2: $S = \tau_1 \tau_2 \tau_3 \tau_1 \tau_2 \tau_3 \tau_2 \tau_1 \tau_4$

We reach $\{(1, 3), (1, 2), (0, 3)\}$ from applying $\tau_1 \tau_2 \tau_3 \tau_1 \tau_2 \tau_3 \tau_2 \tau_1$ ($\tau_1 = \mu_1 \mu_2$, $\tau_2 = \mu_3 \mu_4$, and $\tau_3 = \mu_5 \mu_6$) and then $\tau_4 = \mu_1 \mu_4 \mu_1 \mu_5 \mu_1$ yields $C^S = [\sigma^{-1}C_1^3, C_1^3, C_1^2, \sigma^{-1}C_1^2, \sigma^{-1}C_0^3, C_0^3] = \{(2, -3, 0, 5, -6, 3), (3, -4, 1, 4, -5, 2), (2, -3, 1, 3, -4, 2), (1, -2, 0, 4, -5, 3), (2, -2, -1, 5, -5, 2), (3, -3, 0, 4, -4, 1)\}$. 
Example 2: $S = \tau_1 \tau_2 \tau_3 \tau_1 \tau_2 \tau_3 \tau_2 \tau_1 \tau_4$

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$$[(2, -3, 0, 5, -6, 3), (3, -4, 1, 4, -5, 2), (2, -3, 1, 3, -4, 2), (1, -2, 0, 4, -5, 3), (2, -2, -1, 5, -5, 2), (3, -3, 0, 4, -4, 1)].$$
Example 3:  \( S = \tau_1 \tau_2 \tau_3 \tau_1 \tau_3 \tau_2 \tau_1 \tau_4 \tau_5 \)

\[
[(0, -2, 1, 3, -5, 4), (-1, -1, 0, 4, -6, 5), (0, -1, -1, 5, -6, 4), \\
(1, -2, 0, 4, -5, 3), (0, -1, 0, 3, -4, 3), (-1, 0, -1, 4, -5, 4)].
\]
Example 3: $S = \tau_1 \tau_2 \tau_3 \tau_1 \tau_3 \tau_2 \tau_1 \tau_4 \tau_5$

$$[(0, -2, 1, 3, -5, 4), (-1, -1, 0, 4, -6, 5), (0, -1, -1, 5, -6, 4),
(1, -2, 0, 4, -5, 3), (0, -1, 0, 3, -4, 3), (-1, 0, -1, 4, -5, 4)].$$
Possible Shapes of Aztec Castles

(+,−,+,+,−) (−,−,+,−,−,+) (−,+,−,−,+,−) (−,+,+,−,+,+) (+,+,−,+,+,−) (+,−,−,+,−,−)

(+,−,−,−,+) (−,−,+−,+,−) (−,+,+,−,+) (+,+,−,−,+,−) (+,−,+,+,−,−) (+,−,−,+,+,−)

(+,+,+−,+,−) (+,−,−,−,+) (+,−,+,+,+−,−) (+,−,+,−,+,−) (+,−,−,+,+,−) (+,−,−,+,+,−)
Cross-section when $k$ positive

\[ i + j = k - 1 \]

\[ j = k - 1 \]

\[ i = -k \]
Cross-section when $k$ negative
Self-intersecting Contours

Algebraic formula

\[ z_{i,j}^k = x_r \ A^{\left\lfloor \frac{(i^2 + ij + j^2 + 1) + i + 2j}{3} \right\rfloor} B^{\left\lfloor \frac{(i^2 + ij + j^2 + 1) + 2i + j}{3} \right\rfloor} C^{\left\lfloor \frac{i^2 + ij + j^2 + 1}{3} \right\rfloor} D^{\left\lfloor \frac{(k-1)^2}{4} \right\rfloor} E^{\left\lfloor \frac{k^2}{4} \right\rfloor} \]

still works for \((a, b, c, d, e, f)\) when alternating in signs but combinatorial formula for such cases open.

\((+, -,+,-,+,-)\)

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**Question:** There are many other quivers that arise in the physics literature or admit brane tilings. Can we obtain analogous combinatorial interpretations of toric cluster variables in these cases as well?

**Question:** Finally, we focused on cluster expansions assuming the initial cluster was Model I. What if we start from a different model. It appears that it the initial cluster is of Model IV that one gets Hexagonal dungeons. T. Lai and I plan to do further work relating Dungeons and Dragons.
Thanks for Coming  (Slides at http://math.umn.edu/~musiker/Fargo.pdf)


• *Beyond Aztec Castles: Toric Cascades in the $dP_3$ Quiver* (with Tri Lai), arXiv:1512.00507.